

other physical assets. We will see that in CIA inflation induces agents away from activities that require money, so analogous conclusions will come back.

Given their wealth, agents decide how to allocate it (after consumption) between money and capital. What is money? Their flow of funds is:

$$c_t + K_t + \frac{M_t}{P_t} = f(K_{t-1}, L_t) + (1 - \delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t$$

There is simply a government that prints money M (that is, issues a liability called M). This is OUTSIDE money: the important characteristic of this asset is that is unbacked and has only value as a means of exchange. Clearly, its value is the amount of goods that you can buy with it. Any changes in the real value of this liability:

$$\frac{M_t - M_{t-1}}{P_t} = \text{REVENUE}$$

generate a revenue for the government which then transfers this revenue to economic agents. How do we end up writing these flows of funds? Consider the following table (Mundell was a master in this...) where in the columns we have the markets, in the rows we have the agents.

<i>agents/markets</i>	goods	money	<i>bc</i>
HH	$(-c_t - K_t + f(K_{t-1}, L_t) + (1 - \delta) K_{t-1} + T_t) P_t$	$-M_t + M_{t-1}$	$= 0$
G	$-P_t T_t$	$M_t - M_{t-1}$	$= 0$
<i>equilibrium</i>	$-c_t - K_t + f(K_{t-1}, L_t) + (1 - \delta) K_{t-1} = 0$	0	

Let us now turn back to the households. We assume households consume a constant fraction of their current income:

$$c_t = (1 - s) \left[f(K_{t-1}, L_t) + T_t + \frac{M_{t-1}}{P_t} \right]$$

so what's left is...

$$K_t + \frac{M_t}{P_t} = s \left[f(K_{t-1}, L_t) + \frac{M_{t-1}}{P_t} + T_t \right] + (1 - \delta) K_{t-1}$$

now : allocate between K and $m \equiv M/P$

$$\phi \equiv m/K \quad (\phi \text{ can be fixed or a function of inflation rate})$$

Now consider government policy. We assume a very simple structure in which the government prints money (i.e. issues new liabilities) according to the rule $M_t = \theta_t M_{t-1}$. Then, using government budget constraint:

$$T_t + \frac{M_{t-1}}{P_t} = \frac{M_t}{P_t} = \frac{\theta_t M_{t-1}}{P_t} = \frac{\theta_t M_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} = \frac{\theta_t}{\pi_t} m_{t-1}$$

hence

$$K_t + m_t = s \left[f(K_{t-1}) + \frac{\theta_t}{\pi_t} m_{t-1} \right] + (1 - \delta) K_{t-1}$$

$$\text{using } m_t = \phi K_t$$

$$(1 + \phi) K_t = s \left[f(K_{t-1}) + \frac{\theta_t}{\pi_t} \phi K_{t-1} \right] + (1 - \delta) K_{t-1}$$

The last equation is crucial. From it we can show the following results:

1. when $\phi = 0$, the model boils down to the Solow growth model
2. The solution for the SS capital stock:

$$s f(K^{SS}) = \delta K^{SS} + (1 - s) \phi K^{SS} \quad (\text{x})$$

shows that a higher ϕ reduces capital accumulation

$$K^{SS} = K \left(\frac{-}{\phi} \right)$$

in the Solow model, K is constant when savings are enough to replace depreciated capital. Here when income rises by 1, individual demand more money in real terms by $(1-s)m = (1-s)\phi k$. This is the last term in (x) which shows how money demand lowers capital accumulation.

3. Proportional changes in θ will in steady state affect π in the same proportion (from $\pi_t m_t = \theta_t m_{t-1}$, m is constant only when $\theta = \pi$). Thus if ϕ is constant, money is NEUTRAL in the long-run.
4. However, if ϕ is endogenous (why? in principle you would expect that demand for an asset depends positively on its return)

$$\text{if } \phi = \phi(\pi), \phi' < 0, \text{ then } \uparrow \theta \rightarrow \uparrow \pi \rightarrow \downarrow \phi \rightarrow \uparrow K^{SS}$$

this is the celebrated Mundell-Tobin effect: *Inflation leads to higher accumulation of capital since it shifts resources away from less productive (money) to more productive (capital) assets.* (Mundell argued that inflation could lower the real interest rate permanently as wealth holders rebalance portfolios away from money and reduce consumption)

Assessment of this model:

- (1) no empirical evidence shows that higher output is associated with high money growth and inflation
- (2) there is no role for money in this model
- (3) the behavioral assumptions of the model are *ad hoc*

3.2 Sidrauski model

3.2.1 The setup

We put now money in the utility function. The advantage of doing so is that money is in this way not dominated by bonds that provide and pay an interest rate.

$$\begin{aligned} & \max_{C_t, K_t, M_t/P_t, L_t} E_0 \sum_{t=0}^{\infty} \beta^t u \left(C_t, \frac{M_t}{P_t}, L_t \right) \\ \text{s.t. } & C_t + K_t + \frac{M_t}{P_t} + B_t = R_{t-1} B_{t-1} + f(K_{t-1}, L_t) + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t \\ & Y_t = f(K_{t-1}, L_t) \end{aligned}$$

notice that once you choose K , M/P , and real bonds B , you are automatically choosing C as well. Notice that we could also drop *real bonds* from this formulation since they in equilibrium simply offer the same return as capital and define the gross return on capital as R .

We can think of $\frac{M}{P}$ as the service flow which is provided by money holdings. The budget constraint simply states that given the current income, its assets, and any transfers received by the government T_t , the households allocates its resources between (1) consumption; (2) gross investment in physical capital; gross accumulation of (3) real money and (4) bonds.

The first order conditions for this problem are:

$$u_{C,t} = \beta u_{C,t+1} R_t \tag{a}$$

$$u_{C,t} = \beta u_{C,t+1} [1 - \delta + f_K(K_t)] \tag{b}$$

$$u_{l,t} = u_{C,t} f_l(L_t) \tag{c}$$

$$u_{C,t} = u_{m,t} + \beta u_{C,t+1} \frac{1}{\pi_{t+1}} \left[= u_{m,t} + \frac{\beta}{\pi_{t+1}} u_{m,t+1} + \frac{\beta^2}{\pi_{t+1}\pi_{t+2}} u_{m,t+2} + \frac{\beta^3}{\pi_{t+1}\pi_{t+2}\pi_{t+3}} u_{C,t+3} \right] \tag{d}$$

(a) to (c) are familiar; (d) is the typical expression for the price of an asset: if I give up consumption today and decide to hold money forever from then on, I will enjoy the stream of utility services in square brackets, which will be eroded from the rise in prices between t and the future.

Together with this, we also have appropriate transversality conditions that state:

$$\begin{aligned}\lim_{t \rightarrow \infty} \beta^t u_{C,t} m_t &= 0 \\ \lim_{t \rightarrow \infty} \beta^t u_{C,t} K_t &= 0\end{aligned}$$

Another important issue: suppose you also have nominal bonds Z_t traded offering I_t . Equilibrium requires:

$$\begin{aligned}u_{C,t} &= \beta u_{C,t+1} R_t \\ u_{C,t} \frac{1}{P_t} &= \beta u_{C,t+1} \frac{1}{P_{t+1}} I_t\end{aligned}$$

which implies:

$$R_t = E_t \left(I_t \frac{1}{\pi_{t+1}} \right)$$

which is the Fisher relationship. If we note that these are gross quantities, then we can also write $r_t = i_t - \pi_{t+1}$.

3.2.2 Parametrization

Suppose we parametrize the model as follows (here $m_t \equiv M_t/P_t$ denotes real money):

$$\begin{aligned}u &= \frac{(aC_t^{1-b} + (1-a)m_t^{1-b})^{\frac{1-\phi}{1-b}}}{1-\phi} - \frac{\tau L_t^\eta}{\eta} \\ Y_t &= A_t K_{t-1}^\alpha L_t^{1-\alpha}\end{aligned}$$

When $\phi = b = 1$, this specification becomes $u = a \log c + (1-a) \log m - \dots$

Then, defining with $x_t = aC_t^{1-b} + (1-a)m_t^{1-b}$

$$\begin{aligned}u_c &= a x_t^{\frac{b-\phi}{1-b}} C_t^{-b} \\ u_m &= (1-a) x_t^{\frac{b-\phi}{1-b}} m_t^{-b}\end{aligned}$$

so that the *MRS* between consumption and money becomes (labor does not matter...):

$$MRS_{cm} = \frac{a}{1-a} \left(\frac{C_t}{m_t} \right)^{-b}$$

so that the *EOS* between c and m

$$\frac{d \log (C_t/m_t)}{d \log MRS} = -\frac{1}{b}$$

so that when $b = 0$ consumption and money are perfect substitutes. As b rises, they become more and more complements.

Another important thing to observe. Taking derivative of u_c with respect to m

$$u_{cm} = a x_t^{\frac{b-\phi}{1-b}-1} C_t^{-b} (b-\phi) (1-a) m_t^{-b}$$

which is positive if $b > \phi$. That is if $b > \phi$ consumption and money are complements.

3.2.3 The steady state

3.2.3.1 Monetary side

Changes in money supply are made through transfers to the public. That is

$$T_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t}{P_t} - \frac{M_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t}$$

Steady state transfers are simply:

$$T = \frac{M}{P} - \frac{M}{P} \frac{1}{\Pi} = m \left(\frac{\Pi - 1}{\Pi} \right) = \frac{M}{P} - \frac{M}{\theta P} = m \left(\frac{\theta - 1}{\theta} \right)$$

so that inflation in steady state is a monetary phenomenon.

$$\Pi = \theta$$

For money demand, using the nominal interest rate:

$$u_{C,t} = u_{m,t} + \frac{1}{I_t} u_{C,t}$$

some algebra shows that

$$\begin{aligned} u_c \left(\frac{I-1}{I} \right) &= u_m \\ \frac{a}{1-a} \left(\frac{C}{m} \right)^{-b} &= \frac{I}{I-1} \\ I &= \frac{\theta}{\beta} \\ \frac{a}{1-a} \left(\frac{C}{m} \right)^{-b} &= \frac{\theta}{\theta - \beta} \end{aligned}$$

3.2.3.2 Real side

From (a) we can see that the capital output ratio and the consumption output ratio are not affected by anything that has to do with money. When the marginal utility of consumption is constant, the steady state capital stock solves, assuming $Y = K^\alpha L^{1-\alpha}$:

$$\frac{\alpha Y}{RK} = 1 - \frac{1-\delta}{R}$$

Also:

$$\begin{aligned} \beta R &= 1 \\ \frac{K}{Y} &= \frac{\alpha}{R - (1-\delta)} \\ \frac{C}{Y} &= 1 - \frac{\delta K}{Y} = 1 - \frac{\alpha \delta}{R - (1-\delta)} \end{aligned}$$

One can see that K/Y is independent of all parameters of the utility function; the inflation rate. As in Tobin model, inflation in steady state just equals the growth rate of money supply. Model displays superneutrality of money in steady state: the real equilibrium is independent of the rate of growth of money.

It remains to be seen whether L is independent of money supply in this setup. Using (c):

$$\tau L^\eta = a \left(a C^{1-b} + (1-a) m^{1-b} \right)^{\frac{b-\phi}{1-b}} C^{-b} (1-\alpha) Y$$

unless $b = \phi$, L depends on m , which depends on θ . If $b > \phi$, faster money growth reduces m . Consumption falls (since C and m are complements) and therefore L falls.

As a matter of fact, we do not need to calculate L so long as we are interested in analyzing small deviations from the steady state. All that matters for labor supply is η , which dictates the elasticity of labor supply with respect to the real wage.

3.2.4 The linear approximation

3.2.4.1 The consumption function

The only awkward stuff is how to linearize the u_c and u_m terms.

We know that

$$\begin{aligned} u_{ct} &= ax_t^{\frac{b-\phi}{1-b}} C_t^{-b} \\ u_{mt} &= (1-a)x_t^{\frac{b-\phi}{1-b}} m_t^{-b} \\ \hat{u}_{ct} &= \frac{b-\phi}{1-b} \hat{x}_t - b\hat{C}_t \\ \hat{u}_{mt} &= \frac{b-\phi}{1-b} \hat{x}_t - b\hat{m}_t \end{aligned}$$

Simply from

$$\begin{aligned} x_t &= aC_t^{1-b} + (1-a)m_t^{1-b} \\ \hat{x}_t &= \frac{aC_t^{1-b}}{aC_t^{1-b} + (1-a)m_t^{1-b}} (1-b)\hat{C}_t + \frac{(1-a)m_t^{1-b}}{aC_t^{1-b} + (1-a)m_t^{1-b}} (1-b)\hat{m}_t \\ \hat{x}_t &= \gamma(1-b)\hat{C}_t + (1-\gamma)(1-b)\hat{m}_t \end{aligned}$$

so that the Euler equation for consumption is:

$$\omega_1 (\hat{C}_t - \hat{C}_{t+1}) - \omega_2 (\hat{m}_t - \hat{m}_{t+1}) = -\hat{R}_t$$

where

$$\begin{aligned} \gamma &= \frac{aC_t^{1-b}}{aC_t^{1-b} + (1-a)m_t^{1-b}} = \left(1 + \frac{1-a}{a} \left(\frac{m}{C}\right)^{1-b}\right)^{-1} \\ \omega_1 &= \gamma\phi + (1-\gamma)b \\ \omega_2 &= (b-\phi)(1-\gamma) \end{aligned}$$

3.2.4.2 The money demand

From

$$\begin{aligned} \frac{a}{1-a} \left(\frac{C_t}{m_t}\right)^{-b} &= \frac{I_t}{I_t - 1} \\ -b(\log C_t - \log m_t) &= \log I_t - \log(I_t - 1) \\ \hat{m}_t &= \hat{C}_t - \frac{1}{b(I-1)} \hat{I}_t \end{aligned}$$

3.2.4.3 The labor supply schedule

$$\begin{aligned} u_{l,t} &= u_{C,t} f_l(L_t) \\ (\eta - 1)\hat{L}_t &= -\omega_1 \hat{C}_t + \omega_2 \hat{m}_t + \hat{Y}_t - \hat{L}_t \\ \hat{Y}_t - \omega_1 \hat{C}_t + \omega_2 \hat{m}_t &= \eta \hat{L}_t \end{aligned}$$

3.2.4.4 The evolution of money supply

We assume money balances evolve according to

$$M_t = \theta_t M_{t-1}$$

first to express in variables which are constant in steady state we divide everything by P_t . Next we multiply and divide the RHS by P_{t-1} , so that linearizing brings

$$\widehat{m}_t = \widehat{\theta}_t - \widehat{\pi}_t + \widehat{m}_{t-1}$$

3.2.4.5 The complete model

The log-linear equilibrium will be (here I change notation: the variables denote percentage deviations from the steady state):

$$Y_t = \alpha k_{t-1} + (1 - \alpha) L_t + A_t \quad (1)$$

$$Y_t = \frac{C}{Y} C_t + \frac{K}{Y} (K_t - (1 - \delta) K_{t-1}) \quad (2)$$

$$R_t = \frac{\alpha Y}{Rk} (Y_{t+1} - K_t) \quad (3)$$

$$-R_t = \omega_1 (C_t - C_{t+1}) - \omega_2 (m_t - m_{t+1}) \quad (4)$$

$$\eta L_t = Y_t - \omega_1 C_t + \omega_2 m_t \quad (5)$$

$$m_t = C_t - \frac{1}{b(I-1)} (R_t + \pi_{t+1}) \quad (6)$$

$$m_t - m_{t-1} = -\pi_t + u_t \quad (7)$$

In the special case in which utility is separable in consumption and money balances, $b = \phi$, $\omega_1 = \phi$ and $\omega_2 = 0$. This is an interesting case because now the equations of the model (1) to (7) can be separated in two independent blocks. The first block includes equations (1) to (5) adequately modified:

$$Y_t = \alpha k_{t-1} + (1 - \alpha) L_t + A_t \quad (1)$$

$$Y_t = \frac{C}{Y} C_t + \frac{K}{Y} (K_t - (1 - \delta) K_{t-1}) \quad (2)$$

$$R_t = \left(1 - \frac{1 - \delta}{R}\right) (Y_{t+1} - K_t) \quad (3)$$

$$-R_t = \omega_1 (C_t - C_{t+1}) \quad (4)$$

$$\eta L_t = Y_t - \omega_1 C_t \quad (5)$$

we can see that we can solve for the dynamics of C_t , K_t , Y_t , R_t and L_t independently of the rest of the model. Money is thus completely neutral for the real variables, in and out of the steady state.

Similarly, the equations (6) and (7) to study inflation and money growth independently of the real variables. To this end, fix the real interest rate to some constant, so that consumption will be constant too. Then, from (6)

$$m_t = C_t - \frac{1}{b(I-1)} (R_t + \pi_{t+1})$$

we obtain

$$M_t - p_t = -\frac{1}{b(I-1)} (p_{t+1} - p_t)$$

this is Cagan money demand. If people expect high inflation in the future, they will reduce their real model holdings now. You can solve this equation for p_t

$$p_t = \chi p_{t+1} + (1 - \chi) M_t$$

for some constant χ . Hence the price level today depends on current and future expected money growth (which is described by (7)).

3.2.5 Calibration

We want to calibrate the model so as to get reasonable values for the big real ratios, for the elasticity of money demand to the interest rate, and for the money consumption ratio.

From $\frac{a}{1-a} \left(\frac{C}{m}\right)^{-b} = \frac{\theta}{\theta-\beta}$ and $m_t = C_t - \frac{1}{b(I-1)} (R_t + \pi_{t+1})$ we can see that the elasticity of money demand to the nominal interest rate is $\frac{1}{b(I-1)}$. Assuming $I = \theta/\beta = 1.0125/.99 = 1.023$ and given that this elasticity is in the neighborhood of 3, we need a value of b such that $\frac{1}{b(I-1)} = 3$. Hence $b = 14.5$.

Estimates of $\frac{C}{m}$ depend on which measure of m we use.

In the US: <http://www.federalreserve.gov/releases/H6/hist/h6hist1.txt>:

$$\begin{aligned} M1 &= 1.285 \text{ trillion \$} \\ M2 &= 6.132 \\ M3 &= 8.956 \end{aligned}$$

quarterly consumption is as of 2003Q2 is $7690/4 = 1.92$ trillion \$. Hence if we take a combo of $M1$ and $M2$ we can use $C/m = 1$ which yields

$$a = 0.978$$

This gives us $\gamma = 0.978$. As the intertemporal elasticity of substitution in consumption is $\omega_1 = \gamma\phi + (1-\gamma)b = .978\phi + 14.5(1-.978) = 0.978\phi + 0.319$, a value of $\phi = 2$ gives us $\omega_1 = 2.3$, which is plausible.

3.2.6 Dynamics

- Model generates non-trivial dynamics only when $u_{cm} <> 0$ (marginal utility of consumption falls when you have more money). Remember that

$$\begin{aligned} \text{sign } u_{cm} &= \text{sign } a x_t^{\frac{b-\phi}{1-b}-1} C_t^{-b} (b-\phi) (1-a) m_t^{-b} = \text{sign } (b-\phi) \\ u_{l,t} &= u_{C,t} f_l(L_t) \end{aligned}$$

1. $b > \phi$. $u_{cm} > 0$. When M rises persistently (autocorrelation 0.8), $E_t\pi_{t+1}$ rises, hence m falls, hence u_C falls. Hence consume more leisure ($u_{lt} = L^{\eta-1}$ falls, hence L falls given that $\eta > 1$), and work less. Output falls.
 2. $b < \phi$. $u_{cm} < 0$. When M rises persistently, $E_t\pi_{t+1}$ rises, hence m falls, hence u_C rises, consume less leisure, more work.
- So far, we have talked about persistent rise in M that raises $E_t\pi_{t+1}$. Assume a purely temporary money supply shock. Given that future money growth rates are unaffected, $E_t\pi_{t+1}$ is also unaffected. None of the variables in (1) to (6) is affected. Hence $m_t = 0$. From

$$\begin{aligned} m_t - m_{t-1} &= -\pi_t + u_t \\ 0\% - 0\% &= -1\% + 1\% \end{aligned}$$

we see that inflation rises by 1% in the period of the shock, and then reverts to the baseline.

3.2.7 Welfare losses from inflation

Lucas proposes to estimate the welfare costs of inflation as the % increase in consumption that is required to make the household indifferent between a nominal interest rate of i_1 and a nominal interest rate of 0. Assume labor is supplied inelastically ($\eta = \infty$), so we focus on the consumption-real balances margin only. Look at steady states only:

$$\begin{aligned}
 u(C, m, L) &= \frac{(aC_t^{1-b} + (1-a)m_t^{1-b})^{\frac{1-\phi}{1-b}}}{1-\phi} - \frac{\tau L_t^\eta}{\eta} \\
 m &= C \left(\frac{1-a}{a} \frac{\theta}{\theta-\beta} \right)^{\frac{1}{b}} \text{ in steady state} \\
 u(C, m(C, \theta)) &= u(C, \theta) = \frac{\left(aC^{1-b} + (1-a)C^{\frac{1-b}{b}} \left(\frac{1-a}{a} \frac{\theta}{\theta-\beta} \right)^{\frac{1-b}{b}} \right)^{\frac{1-\phi}{1-b}}}{1-\phi}
 \end{aligned}$$

The welfare loss from any positive inflation rate $\bar{\theta} > 1$ is the value of C that solves

$$\bar{u} = u(1, 1) = u(\bar{C}, \bar{\theta})$$

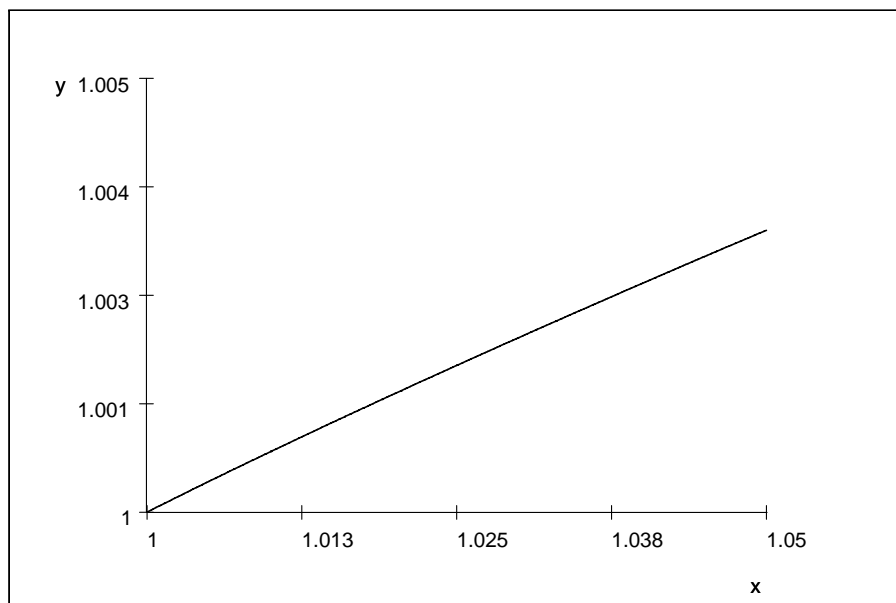
The value of \bar{C} that solves the expression above is the compensation needed to be as well off with an interest rate $\bar{\theta}$ as with 1. Note that \bar{C} is the % change in C require to compensate for the higher nominal interest rate.

Some numerical examples. Suppose we start from a steady state in which $\theta = 1$. Assume same parameters as above. The utility achieved when $C = 1$ and $\theta = 1$ is equal to

$$\begin{aligned}
 (1-\phi)u(C=1, \theta=1)^{\frac{1-b}{1-\phi}} &= .978C^{1-14.5} + .022C^{\frac{1-14.5}{14.5}} \left(\frac{.022}{.978} \frac{\theta}{\theta-.99} \right)^{\frac{1-14.5}{14.5}} \\
 \bar{u} &= .978 + .022 \left(\frac{.022}{.978} \frac{1}{1-.99} \right)^{\frac{-13.5}{14.5}} = 0.98834
 \end{aligned}$$

Hence we look for all other (C, θ) pairs such that utility is equal to \bar{u} . The solution to this equations are in the graph, where y is \bar{C} and θ is the steady state inflation rate.

$$.98834 = .978y^{1-14.5} + .022y^{\frac{1-14.5}{14.5}} \left(\frac{.022}{.978} \frac{x}{x-.99} \right)^{\frac{1-14.5}{14.5}}$$



Consumption (y) required to keep utility constant for every level of the nominal interest rate (x)

A quarterly inflation rate of 2.5% (10% per annum) would require an increase in steady state consumption of 0.2% per quarter (0.8% at an annual rate). Lucas obtains slightly higher numbers (1%), but the substance is the same.

3.2.8 The optimal rate of inflation

Marginal benefit of money is minus the nominal interest rate, marginal cost is zero, want the nominal rate to be zero. It is also known as the optimum quantity of money. Consider the following simple problem:

$$\begin{aligned} & \max_m u(C, m) \\ \text{s.t. } C &= Y - \delta K \\ & \rightarrow u_m = 0 \rightarrow I_t = 1 \text{ and } R_t = \frac{1}{\pi_{t+1}} \end{aligned}$$

The idea is that the optimal rate of inflation is a rate of deflation $\frac{P_t}{P_{t+1}}$ approximately equal to the real return on capital. This result is known as the Friedman rule.

3.3 Cash in advance model

We want to construct a model in which money is used as a medium of exchange that facilitates transactions yielding utility indirectly. In other words, money is needed to make transactions.

$$\begin{aligned} & \max_{K_t, M_t/P_t, L_t, Z_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \\ \text{s.t. } C_t + K_t + \frac{M_t}{P_t} + \frac{Z_t}{P_t} &= \frac{P_{t-1}}{P_t} I_{t-1} \frac{Z_{t-1}}{P_{t-1}} + f(K_{t-1}, L_t) + (1 - \delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t \\ Y_t &= f(K_{t-1}, L_t) \\ C_t &= \frac{M_{t-1}}{P_t} + T_t \quad [\mu_t] \end{aligned} \tag{CIA_GA}$$

TIMING DIGRESSION: Purchases of consumption goods require now cash in hand M_{t-1} : idea is that the agent enters the period with money holdings M_{t-1} and receives a lump-sum transfer T_t in real terms. The

timing is the goods market opens first. The household cannot adjust its portfolio holdings after the shocks hit. Another possibility is that households can adjust their portfolios after the shocks hit.

$$C_t + \frac{Z_t}{P_t} = \frac{M_{t-1}}{P_t} + T_t \quad (\text{CIA_AG})$$

here after the transfer you can still choose your asset position before deciding your consumption level. Hence the asset market first, then the goods market.

Notice that Z_{t-1} are the holdings of a nominal bond that yields a gross return of I_{t-1} from period $t-1$ to t (unlike the real bond B_t).

The Lagrangian for this problem is (replacing C from the flow of funds equation)

$$\begin{aligned} l = & u \left(\frac{I_{t-1}P_{t-1}}{P_t} \frac{Z_{t-1}}{P_{t-1}} + f(K_{t-1}, L_t) + (1-\delta)K_{t-1} + \frac{M_{t-1}}{P_t} + T_t - K_t - \frac{M_t}{P_t} - \frac{Z_t}{P_t}, L_t \right) + \\ & \beta u \left(\frac{I_t P_t}{P_{t+1}} \frac{Z_t}{P_t} + f(K_t, L_{t+1}) + (1-\delta)K_t + \frac{M_t}{P_t} \frac{P_t}{P_{t+1}} + T_{t+1} - K_{t+1} - \frac{M_{t+1}}{P_{t+1}} - \frac{Z_{t+1}}{P_{t+1}}, L_{t+1} \right) \\ & - \mu_t \times \left(\frac{I_{t-1}P_{t-1}}{P_t} \frac{Z_{t-1}}{P_{t-1}} + f(K_{t-1}, L_t) + (1-\delta)K_{t-1} - K_t - \frac{M_t}{P_t} - \frac{Z_t}{P_t} \right) \\ & - \beta \mu_{t+1} \times \left(\frac{I_t P_t}{P_{t+1}} \frac{Z_t}{P_t} + f(K_t, L_{t+1}) + (1-\delta)K_t - K_{t+1} - \frac{M_{t+1}}{P_{t+1}} - \frac{Z_{t+1}}{P_{t+1}} \right) \end{aligned}$$

The first order conditions are, choosing Z_t/P_t , K_t , M_t/P_t and L_t

$$u_{C,t} - \mu_t = \beta \frac{P_t}{P_{t+1}} I_t (u_{C,t+1} - \mu_{t+1}) \quad (1)$$

$$u_{C,t} - \mu_t = \beta (u_{C,t+1} - \mu_{t+1}) [1 - \delta + f_K(K_t)] \quad (2)$$

$$u_{C,t} - \mu_t = \beta u_{C,t+1} \frac{P_t}{P_{t+1}} \quad (3)$$

$$u_{C,t} - \mu_t = \frac{u_{L,t}}{f_L(L_t)} \quad (4)$$

One can use Euler equations for C and for M/P in several ways. Solving for the value of money ($1/P_t$) from equation (3)

$$\begin{aligned} u_{C,t} &= \mu_t + \beta u_{C,t+1} \frac{P_t}{P_{t+1}} = \mu_t + \beta \frac{P_t}{P_{t+1}} \left(\mu_{t+1} + \beta u_{C,t+2} \frac{P_{t+1}}{P_{t+2}} \right) = \mu_t + P_t \left(\beta \frac{\mu_{t+1}}{P_{t+1}} + \beta^2 \frac{\mu_{t+2}}{P_{t+2}} + \dots \right) \\ \frac{1}{P_t} &= \frac{\sum_{i=1}^{\infty} \beta^i \frac{\mu_{t+i}}{P_{t+i}}}{u_{C,t} - \mu_t} \end{aligned} \quad (5)$$

whereas the nominal interest rate will equal, combining (1) and (3):

$$\begin{aligned} \beta u_{C,t+1} \frac{P_t}{P_{t+1}} &= \beta \frac{P_t}{P_{t+1}} I_t (u_{C,t+1} - \mu_{t+1}) \\ I_t &= \frac{u_{C,t+1}}{u_{C,t+1} - \mu_{t+1}} = 1 + \frac{\mu_{t+1}}{u_{C,t+1} - \mu_{t+1}} \end{aligned} \quad (6)$$

- The first expression (5) says that the value of money (relative to the marginal utility of consumption) is equal to the present value of the marginal utility of money in all future periods. Money is valuable so long as it provides utility services, i.e. $\mu > 0$.
- Similarly, the nominal interest rate is positive in (6) so long as $\mu_{t+1} > 0$.
- Looking at the consumption / leisure trade-off in (4), we can see that, whenever $\mu > 0$, $u_{C,t} > u_{L,t}$. Consumption is therefore lower (and labor supply lower) given the distortion towards leisure that the CIA induces. Thus in a CIA model a positive $I_t - 1$ acts as a tax on consumption, raising the price of consumption above its production cost. This is the sense in which in the CIA model both leisure and investment can be thought of as credit goods, since they are not subject to the cash-in-advance constraint.

3.3.1 Cooley and Hansen's (AER 1989) stochastic CIA model

To analyse the model, we parametrize it in the following way:

$$\begin{aligned} u &= \frac{C_t^{1-\phi}}{1-\phi} - \frac{\tau L_t^\eta}{\eta} \\ Y_t &= A_t K_{t-1}^\alpha L_t^{1-\alpha} \end{aligned}$$

3.3.2 The steady state

From 2 derive steady state K/Y :

$$\begin{aligned} u_{C,t} - \mu_t &= \beta (u_{C,t+1} - \mu_{t+1}) [1 - \delta + f_K(K_t)] \\ \frac{K}{Y} &= \frac{\alpha}{R - (1 - \delta)} \end{aligned}$$

Then from the economywide constraint $Y = C + \delta K$ derive C/Y . Then we want the levels. Need L . From equation (4):

$$C^{-\phi} = \mu + \frac{\tau L^\eta}{(1 - \alpha)Y}$$

From 3

$$C^{-\phi} \left(1 - \frac{\beta}{\theta}\right) = \mu$$

As always, $\Pi = \theta$; Use this result and combine the previous two equations:

$$\begin{aligned} C^{-\phi} \frac{\beta}{\theta} &= \frac{\tau L^\eta}{(1 - \alpha)Y} \\ \frac{C}{Y} &= 1 - \delta \frac{K}{Y} \rightarrow \frac{C^{-\phi} \beta}{Y^{-\phi} \theta} = \frac{\tau L^\eta}{(1 - \alpha)Y^{1-\phi}} \rightarrow \gamma^{-\phi} = \frac{\theta}{\beta} \frac{\tau L^\eta}{(1 - \alpha)Y^{1-\phi}} \\ Y &= AK^\alpha L^{1-\alpha} \rightarrow Y = A^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R - (1 - \delta)}\right)^{\frac{\alpha}{1-\alpha}} L = \chi L \\ L &= L(\theta), L_\theta < 0 \end{aligned}$$

which shows that labor supply depends negatively on the inflation rate. Again, welfare costs of inflation can be calculated by noting that utility depends on C and L , both of which are a function of inflation.

3.3.3 The linearized model

The log-linear equilibrium will be - here variables with a time subscript denote percentage deviations of that variable from the steady state. (denoting with $R = 1/\beta$ the steady state real interest rate) (you have to play a little with the algebra of log-linearizations to get some of the equations, see Walsh textbook).

Note: Walsh uses a slightly different version, he defines $\lambda_t = u_{Ct} - \mu_t$ as the marginal utility of one wealth): it follows that in log-linear terms.

$$\lambda_t = -\phi \Pi R C_t - (\Pi R - 1) \mu_t$$

$$Y_t = \alpha K_{t-1} + (1 - \alpha) L_t + A_t \quad (7)$$

$$Y_t = \frac{C}{Y} C_t + \frac{K}{Y} (K_t - (1 - \delta) K_{t-1}) \quad (8)$$

$$R_t = \left(1 - \frac{1 - \delta}{R}\right) (Y_{t+1} - K_t) \quad (9)$$

$$R_t = \phi R \Pi (C_{t+1} - C_t) + (R \Pi - 1) (\mu_{t+1} - \mu_t) \quad (10)$$

$$\eta L_t = Y_t - \phi R \Pi C_t - (R \Pi - 1) \mu_t \quad (11)$$

$$(R \Pi - 1) \mu_t = -\phi R \Pi C_t + \phi C_{t+1} + \Pi_{t+1} \quad (12)$$

$$C_t - C_{t-1} = -\pi_t + u_t \quad (13)$$

Equation (7) is the production function.

(8) is the definition of output.

(9) defines the return on a real bond (the real interest rate): we don't have explicitly a real bond in the formulation of the problem, but if we add one we would get that the real interest rate equals in levels: $R_t = \frac{P_t}{P_{t+1}} I_t$, in logs $R_t = -\pi_{t+1} + I_t$. We obtain 9 by combining 1 and 2.

(10) is the Euler equation for consumption, derived from equation 2.

(11) is labour supply.

(12) is the money demand equation. It says that the multiplier on the CIA constraint becomes lower if you consume more today (as you consume more, the marginal utility of consumption falls....) and becomes tighter if expected inflation rises.

(13) is the money supply rule. Remember that in the steady state in which the CIA is binding: $C_t = \frac{M_t}{P_t}$

3.3.4 Dynamics

What we said above explains the monetary transmission mechanism in the CIA model: a higher money growth, by raising inflation, shifts demand away from the consumption good towards the credit goods (leisure and investment). Inflation hence reduces steady state labor supply.

Even when $\phi = 1$ and $\Pi = 1$ there is interdependence between real and monetary factors. Combining (11) and (12) and forwarding (13) one period ahead:

$$\begin{aligned} \eta L_t - Y_t &= -E_t C_{t+1} - E_t \Pi_{t+1} \\ E_t C_{t+1} - C_t &= E_t u_{t+1} - E_t \pi_{t+1} \end{aligned}$$

combine the above expressions to obtain:

$$Y_t - \eta L_t - C_t = E_t u_{t+1}$$

hence changes in $E u_{t+1}$ will affect C , L and Y (compare this with the log-log case of Sidrauski's model). In particular, for the case with no capital, $Y = C$ and increases in expected inflation reduce labor supply by a factor which is equal to $1/\eta$.

Comparing the CIA with the MIU reveals that a money shock has a much larger impact on output.

Read the Cooley and Hansen 1989 paper. Why do you think they use the indivisible labor assumption? What are the main testable implications of their model? How does inflation affect welfare? Use the `cia.m` and `cia_go.m` files on my webpage and do the following: (1) what if labor supply elasticity changes? (2) what if the capital share changes? (3) what if the persistence of the money shock changes?

