Chapter 6

Optimal Monetary Policy

Models of monetary economics typically involve as much as four distortions.

1. The first distortion derives from the agents' desire to hold money, given the transaction services that money provides. Since the private cost of holding money is R whereas the social cost of producing it is 0 efficiency would require that the two are set equal by having $R = 0$. Since the real rate is $RR = R - \pi$, this requires inflation to equal minus the real interest rate, that is a steady decline in the price level.

2. The second distortion derives from imperfect competition in the goods market. When $\alpha = 1$ (no capital), from

$$Y = \frac{A^{\frac{\eta}{\rho + \eta - 1}}}{X^{\frac{1}{\rho + \eta - 1}}} < Y^* = A^{\frac{\eta}{\rho + \eta - 1}}$$

   we can see that output will be inefficiently low whenever $X > 1$. To correct this distortion, inflation should be permanently above zero (from the Phillips curve), and this creates in itself a trade-off with the objective in (1)

3. (related to nominal rigidity): firms' inability to adjust prices at any point in time creates a *dynamic markup distortion*. Markups will fluctuate over time around their constant frictionless level.

4. (related to nominal rigidity): even in absence of average inflation ($\pi = 0$), the lack of synchronization in price adjustments will imply the coexistence of different prices for goods that enter symmetrically agents' utility functions and which have a one-to-one marginal rate of transformation. This is a *static markup distortion*.

   Modern models normally deal with (3) and (4). Both distortions can be corrected at once by a zero inflation policy. It is in this context, and starting from the utility function of the representative individual producer, that Rotemberg and Woodford (1997, NBER Macro Annual) show that the period utility loss resulting from deviations from the $X = 1$ allocation can by approximated by a quadratic equation of the form:

$$L_t = \frac{u_c C}{2} \left(\sigma_\pi^2 + \frac{\lambda}{\varepsilon} \frac{\rho + \eta - 1}{1 + \varepsilon (\eta - 1)} \sigma_x^2 \right)$$

where $\lambda = \frac{\theta}{(1-\theta)(1-\beta\theta)}$ is the slope of the Phillips curve[1]

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[1] The values in the first version of the paper for the relative weights were slightly different. The reason is explained in footnote 23 of the Woodford book, page 400.

6.1   Deriving the welfare function

Woodford derives a quadratic loss function that represents a second-order Taylor series approximation to the level of expected utility of the representative household in the rational expectations equilibrium associated with a given policy.

I provide a sketch of the derivation. Appendix 11.6 in Walsh provides a good derivation, as well as Woodford's book. Remember that the period utility function for the representative individual is, ignoring the real balances term and remembering that in equilibrium $Y = C$,

$$\frac{Y_t^{1-\rho}}{1-\rho} - \frac{1}{\eta}(L_t)^{\eta}$$

in a steady state without distortions, when $\alpha = 1$:

$$
\begin{aligned}
Y &= AL & \text{(production function)} \\
wY^{-\rho} &= L^{\eta-1} & \text{(labor supply)} \\
A &= w & \text{(labor demand)} \\
Y^{1-\rho} &= L^{\eta} & \text{(equilibrium)}
\end{aligned}
$$

Next take a Taylor approximation

6.1.1   LHS

$$u(Y) = \frac{Y^{1-\rho}}{1-\rho} + Y^{-\rho}(Y_t - Y) - \rho\frac{Y^{-\rho-1}}{2}(Y_t - Y)^2$$

define now

$$\widetilde{Y} = \frac{Y_t - Y}{Y}$$

$$\widehat{Y}_t = \log(Y_t/Y) = \underbrace{\log(Y/Y)}_{f(\overline{x})} + \underbrace{\frac{1}{f'(\overline{x})}}\underbrace{\frac{Y_t - Y}{Y}}_{x-\overline{x}} - \underbrace{\frac{1}{2}}_{f''(\overline{x})/2}\left(\frac{Y_t - Y}{Y}\right)^2$$

so that

$$\widetilde{Y} = \frac{Y_t - Y}{Y} = \widehat{Y} + \frac{1}{2}\widehat{Y}^2$$

and (drop out constant terms):

$$u(Y) = Y^{1-\rho}\left(\widehat{Y}_t + \frac{1-\rho}{2}\widehat{Y}_t^2\right) \tag{a}$$

6.1.2   RHS

Remember that L is the integral of labor supplied by all households in the economy $\int y(z)\,dz$, and each household on the segment produces good $Y(z) = y$ for notational simplicity. Taylor series of $v(l)$

$$v(l) = l^{\eta}\left(\widehat{l}_t + \frac{\eta}{2}\widehat{l}_t^2\right)$$

integrate wrt z across all households, use $Y^{1-\rho} = l^{\eta}$

$$\int v(l)\,dz = Y^{1-\rho}\left(E\widehat{y}_t + \frac{\eta}{2}E\widehat{y}_t^2\right) = Y^{1-\rho}\left(E\widehat{y}_t + \frac{\eta}{2}\left((E\widehat{y}_t)^2 + VAR(\widehat{y}_t)\right)\right)$$

Use:

$$
\begin{aligned}
Y_t &= \left[\int_0^1 Y_t(z)^{\frac{\varepsilon-1}{\varepsilon}}\,dz\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\widehat{Y}_t &= E\widehat{y}_t + \frac{\varepsilon-1}{\varepsilon}\frac{1}{2}VAR(\widehat{y}_t)
\end{aligned}
$$

to write:

$$\int v\left(l\right)dz = Y^{1-\rho}\left(\widehat{Y}_t - \frac{\varepsilon-1}{2\varepsilon}VAR\left(\widehat{y_t}\right) + \frac{\eta}{2}\underbrace{\left(\widehat{Y}_t^2 + ... + VAR\left(\widehat{y_t}\right)\right)}_{\text{drop } VAR^2 \text{ which is 4th order}}\right) \tag{b}$$

6.1.3   Putting things together

Subtract $(a) - (b)$ and check the relative weight on Y_t^2 (dispersion of average output) versus $VAR\left(y\right)$ (dispersion of individual output produced by each intermediate goods producer/worker).

$$\text{weight on } Y^2 \quad \propto \quad \frac{1-\rho}{2} - \frac{\eta}{2} \propto \rho + \eta - 1$$

$$\text{weight on } VAR\left(y\right) \quad \propto \quad \frac{\varepsilon-1}{2\varepsilon} - \frac{\eta}{2} \propto \eta - \frac{\varepsilon-1}{\varepsilon}$$

the ratio

$$\frac{VAR\left(y\right)}{Y^2} = \frac{\eta - \frac{\varepsilon-1}{\varepsilon}}{\rho + \eta - 1}$$

represents the relative weight on producer's output dispersion in society's welfare.

The term on Y^2 represents deviations of output from its flexible price level, hence is tantamount to X.

The variance of y, individual output produced, instead is related to the variance of the prices that producers face through the individual demand curve for each product.

$$y^* = \left(p^*/P\right)^{-\varepsilon}Y$$
$$VAR\left(y\right) = \varepsilon^2 VAR\left(p^*\right)$$

The variance of prices is turn linked to the inflation rate through

$$P_t = \theta P_{t-1} + \left(1-\theta\right)p^*$$

which can be used to show that, defining weights on inflation and output gap

$$w\left(p^*\right) = \frac{\theta}{1-\theta}\frac{1}{1-\theta\beta}w\left(\pi\right) = \frac{1}{\lambda}w\left(\pi\right)$$

and therefore the relative weight on inflation variance with respect to output variance must satisfy:

$$w\left(\pi_t\right) = \lambda w\left(p^*\right) = \frac{\lambda}{\varepsilon^2}w\left(y\right) = \frac{\lambda}{\varepsilon^2}\frac{\rho + \eta - 1}{\eta - \frac{\varepsilon-1}{\varepsilon}}w\left(X_t\right) = \frac{\lambda}{\varepsilon}\frac{\rho + \eta - 1}{1 + \varepsilon\left(\eta - 1\right)}$$

cfr Walsh page 555 and Woodford page 400.

To sum up one can relate

- $var\left(y\right)$ to the inflation variance times a term which depends on ε and θ. So inflation variability captures the dispersion of output levels across producers of different goods. Inflation concerns become more important the greater price rigidities and the less substitutable goods are.

- Y^2 captures the variability of output around its natural rate

What is a plausible value for $\frac{\lambda}{\varepsilon}\frac{\rho+\eta-1}{1+\varepsilon(\eta-1)}$, the relative weight on output (gap) stabilization? Assume

$$
\begin{aligned}
\theta &= .75 \\
\beta &= .99 \\
\lambda &= \frac{(1-\theta)(1-\theta\beta)}{\theta} \\
\eta &= 1.1 \\
\varepsilon &= 10 \\
\rho &= 1 \\
\xi &= \frac{\lambda}{\varepsilon}\frac{\rho+\eta-1}{1+\varepsilon(\eta-1)}
\end{aligned}
$$

Then $\xi = 4.7208 \times 10^{-3}$. If we measure output gap in % terms and inflation as an annualized inflation rate, then the appropriate weight on x^2 relative to $(4\pi)^2$ becomes $16\xi = .075$, which is much lower than the values of 1 typically assumed in the literature, on grounds such as "equal weight to the two objectives". In other words, the distortions associated with inflation are far greater than those associated with variation in the output gap.

The losses from inflation can be completely elimated by a zero inflation policy. That is, the price level distortion are minimized by creating an environment in which

1.   • those who choose a new price set the old price

     • if so, then the average price level never changes

     • eventually all good prices are the same

     • hence price stability is a sufficient condition for the absence of price dispersion

6.2   Optimal policy

Back to our earlier model. Aggregate demand and supply are given by:

$$
\begin{aligned}
x_t - E_t x_{t+1} + \phi(R_t - E_t\pi_{t+1}) - g_t &= 0 \\
\pi_t - \lambda x_t - \beta E_t\pi_{t+1} - u_t &= 0
\end{aligned}
$$

Consider the problem of a central bank:

$$
\max W = -\frac{1}{2}E_t\left[\sum_{t=0}^{\infty}\beta^t L_t\right]
$$

where:

$$
L_t = \pi_t^2 + \alpha x_t^2
$$

subject to AD and AS above.

Let me spend a few words about this problem: macroeconomics is full of problems in which the return function is quadratic and the constraint is linear. However not all constraints are the same: in the standard problem, the constraints are linear and static. More complicated are cases when the constraint is a low a motion over time. Even more complicated are problems where the constraints involve expectations of future variables, rendering the dynamic programming principle invalid: in this class of problems, in fact, target variables depend not only on policy but also on future expected policy.

Remark 7 *The optimal equilibrium is the one that achieves the lowest possible value of the loss measure W above.*

6.3   Optimal policy under discretion

Under discretion, the central bank expects itself to reoptimize at each successive date, and is unable to commit itself to future paths for inflation and the output gap. That makes the problem rather easy to solve.

Easy way to solve the problem (since we choose x π and R) is first to solve under AS constraint only and then to work out optimal R implied by the aggregate demand curve.

$$\max_{\pi,x} -\frac{1}{2}\left(\pi_t^2 + \alpha x_t^2\right)$$

$$\text{s.t. } \pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t$$

where $f_t \equiv \beta E_t \pi_{t+1} + u_t$ is taken as given.

The problem then becomes:

$$\max_{\pi} -\frac{1}{2}\left(\pi_t^2 + \alpha \left(\frac{\pi_t - f_t}{\lambda}\right)^2\right)$$

yielding (since $\frac{1}{\lambda}(\pi_t - f_t) = x_t$)

$$\pi_t + \frac{\alpha}{\lambda^2}(\pi_t - f_t) = 0 \Rightarrow x_t^b = -\frac{\lambda}{\alpha}\pi_t^b$$

where the superscript b indicates that this is the solution under discretion.

To solve the problem now combine this equilibrium condition with the AS curve

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t$$

and impose that expectations are rational. You will get:

$$\pi_t = \frac{\alpha\beta}{\alpha + \lambda^2}E_t \pi_{t+1} + \frac{\alpha}{\alpha + \lambda^2}u_t \equiv c\pi_{t+1} + du_t$$

This equation can be solved forward to obtain (under $u_{t+1} = \rho u_t$):

$$\begin{aligned}
\pi_t^b &= c\left(c\pi_{t+2} + du_{t+1}\right) + du_t = \\
&= c\left(c\pi_{t+2} + d\rho u_t\right) + du_t = \\
&= d\left(u_t + c\rho u_t + c^2\rho^2 u_t + ...\right) = \frac{d}{1-c\rho}u_t = \\
\pi_t^b &= \frac{\frac{\alpha}{\alpha+\lambda^2}}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}}u_t = \frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}u_t \\
x_t^b &= -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}u_t
\end{aligned}$$

as $\pi_t = wu_t$, $E\pi_{t+1} = w\rho u_t = \rho\pi_t$.

Go back to IS, solved for R_t:

$$\begin{aligned}
\phi R_t &= E_t x_{t+1} + \phi E_t \pi_{t+1} - x_t + g_t \\
\phi R_t &= -E_t \frac{\lambda}{\alpha}\pi_{t+1} + \phi E_t \pi_{t+1} + \frac{\lambda}{\alpha}\pi_t + g_t \\
\phi R_t &= \left(-\frac{\lambda}{\alpha} + \phi + \frac{\lambda}{\alpha\rho}\right)E_t \pi_{t+1} + g_t \\
R_t &= \left(1 + \frac{\lambda(1-\rho)}{\alpha\rho\phi}\right)E_t \pi_{t+1} + \frac{1}{\phi}g_t
\end{aligned}$$

Remark 8 *Optimal policy responds more than one for one to changes in expected inflation.*

6.3.1   Monetary policy trade-offs under discretion

We have found that:

$$-\frac{\alpha}{\lambda}x_t \;\; = \;\; \pi_t = \frac{\alpha}{\alpha\left(1 - \beta\rho\right) + \lambda^2}u_t$$

$$\sigma_\pi^2 \;\; = \;\; \left(\frac{\alpha}{\alpha\left(1 - \beta\rho\right) + \lambda^2}\right)^2 \sigma_u^2$$

$$\sigma_x^2 \;\; = \;\; \left(\frac{\lambda}{\alpha\left(1 - \beta\rho\right) + \lambda^2}\right)^2 \sigma_u^2$$

$$\sigma_x^2 \;\; = \;\; \frac{\lambda^2}{\alpha^2}\sigma_\pi^2$$

Remark 9 α *defines the policymaker preferences. For given value of λ, the last equation describes an inverse relationship (Taylor curve) between the two policy objectives.*

6.4   Commitment

Literature often divided into two strands. The 1980s literature assumes that the output gap goal is to push output permanently above its natural rate. In the Nash equilibrium, this generates inflation with little output gains. The modern literature considers other issues, i.e. optimal rules.

6.4.1   The classic Inflationary Bias Problem

Assume demand and supply are given by:

$$x_t - E_t x_{t+1} + \phi\left[R_t - E_t \pi_{t+1}\right] - g_t \;\; = \;\; 0$$
$$\pi_t - \lambda x_t - \beta E_t \pi_{t+1} - u_t \;\; = \;\; 0$$

Problem is:

$$\max W = -\frac{1}{2}E_t\left[\sum_{t=0}^{\infty}\beta^t L_t\right]$$

where:

$$L_t = \pi_t^2 + \alpha\left(x_t - k\right)^2$$

$k > 0$ reflects the presence of distortions so that socially efficient output exceeds natural level. In this case a discretionary central bank faces the following problem:

$$\max_\pi -\frac{1}{2}\left(\pi_t^2 + \alpha\left(\frac{\pi_t - f_t}{\lambda} - k\right)^2\right)$$

yielding:

$$\pi_t + \frac{\alpha}{\lambda}\left(\frac{\pi_t - f_t}{\lambda} - k\right) \;\; = \;\; 0$$
$$\pi_t + \frac{\alpha}{\lambda}\left(x_t - k\right) \;\; = \;\; 0$$
$$\Rightarrow \;\; x_t = -\frac{\lambda}{\alpha}\pi_t + k$$

To solve the problem now combine this equilibrium condition with the AS curve

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t = \frac{\lambda\alpha}{\alpha + \lambda^2}k + \beta\pi_{t+1} + u_t$$

and impose that expectations are rational. You will get:

$$\pi_t^k = \pi_t^b + \frac{\alpha\lambda}{\alpha(1-\beta)+\lambda^2}k$$

hence inflation equals inflation under the baseline case plus a term related to k. Solving for x_t and remembering that $x_t^b = -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}u_t$:

$$
\begin{aligned}
x_t^k &= -\frac{\lambda}{\alpha}\left(\pi_t^b + \frac{\alpha\lambda}{\alpha(1-\beta)+\lambda^2}k\right) + k \\
&= -\frac{\lambda}{\alpha}\pi_t^b - \frac{\lambda^2}{\alpha(1-\beta)+\lambda^2}k + k = \\
&= x_t^b + \frac{\alpha(1-\beta)}{\alpha(1-\beta)+\lambda^2}k
\end{aligned}
$$

Remark 10 *Under discretionary policy, Inflation is higher and output is slightly above natural level. However the gain disappears as $\beta \to 1$*

6.4.2 The gains from commitment (even) when $k = 0$: the optimum within simple rules

Here we return to our earlier model but we take into account the possibility that central bank actions might affect private agent expectations. However the solution to this simple problem is not as simple as it might look like....

Consider a rule for the target x of the following form:

$$x_t^c = -\omega u_t$$

this corresponds to the rule under discretion x_t^b whenever the central bank chooses $\omega = \frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}$. Under such a rule inflation is:

$$\pi_t^c = u_t + kx_t^c + \beta E_t\pi_{t+1} = u_t(1-\lambda\omega) + \beta E_t\pi_{t+1} = \frac{1-\lambda\omega}{1-\beta\rho}u_t$$

or differently:

$$\pi_t^c = \frac{1}{1-\beta\rho}(u_t + \lambda x_t^c)$$

comparing this with the solution to the discretionary case:

$$\pi_t^b = \lambda x_t^b + \overline{\beta E_t\pi_{t+1}} + u_t$$

when the central bank is unable to manipulate expectations, the scale of trade-off is different. Reducing x_t by 1% reduces π_t^b by λ%, rather than $\frac{\lambda}{1-\beta\rho}$% as in the commitment case.

What is the optimal value of the feedback parameter? Since both π^c and x^c are multiples of u_t, one can

write the L as a function of π and x only.

$$
\begin{aligned}
\max W &= -\frac{1}{2}E_t\left(L_t + \beta L_{t+1} + \beta^2 L_{t+2} + ...\right) \\
\max W &= -\frac{1}{2}E_t\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2 u_t^2 + \alpha\omega^2 u_t^2 + \beta L_{t+1} + \beta^2 L_{t+2} + ...\right) = \\
\max_\omega W &= -\frac{1}{2}\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2\left(1+\rho^2+\rho^4+...\right)u_t^2 + \alpha\left(\omega^2\left(1+\rho^2+\rho^4+...\right)\right)u_t^2\right) = \\
\max_\omega W &= -\frac{1}{2(1-\rho^2)}\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2 u_t^2 + \alpha\omega^2 u_t^2\right) \\
&\Leftrightarrow \frac{\lambda(1-\lambda\omega)}{(1-\beta\rho)^2} = \alpha\omega \\
\omega^c &= \frac{\lambda}{\lambda^2 + \alpha(1-\beta\rho)^2}
\end{aligned}
$$

Given the optimal ω^c, the link between x and π is given by:

$$
\begin{aligned}
x_t^c &= -\omega^c u_t \\
\pi_t^c &= \frac{1-\lambda\omega^c}{1-\beta\rho}u_t = -\frac{1-\lambda\omega^c}{1-\beta\rho}\frac{1}{\omega^c}x_t^c = -\frac{\alpha(1-\beta\rho)}{\lambda}x_t^c \\
x_t^c &= -\frac{\lambda}{\alpha(1-\beta\rho)}\pi_t^c = -\frac{\lambda}{\alpha_c}\pi_t^c
\end{aligned}
$$

where $\alpha_c = \alpha(1-\beta\rho) < \alpha$ implies that commitment allows the authority to face a better trade-off. Put differently, under commitment for given rise in inflation the output gap has to fall less $(1-\beta\rho) < 1$. The monetary policy authority can be more aggressive against inflation.

To sum up, these are the outcomes that we obtain under the different assumptions about central bank behavior. Once demand shocks are offset, central bank can achieve a better outcome by responding more aggressively to inflation. That is why under commitment the central bank faces a better trade-off.

		π	x	trade-off	int.rate R_t^*
discr, no π bias		$\pi_t^b = \frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}u_t$	$x^b = -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}u_t$	$\sigma_x^2 = \frac{\lambda^2}{\alpha^2}\sigma_\pi^2$	$\left(1+\frac{\lambda(1-\rho)}{\alpha\rho\phi}\right)\pi_{t+1} + \frac{g_t}{\phi}$
discr, π bias		$\pi^k = \frac{\alpha(u_t+\lambda k)}{\alpha(1-\beta\rho)+\lambda^2}$	$x^k = \frac{\alpha(1-\beta)k}{\alpha(1-\beta)+\lambda^2} - \frac{\lambda}{\alpha_c+\lambda^2}u_t$		
commitment		$\pi^c = \frac{\alpha(1-\beta\rho)}{\alpha(1-\beta\rho)^2+\lambda^2}u_t$	$x^c = -\frac{\lambda}{\alpha(1-\beta\rho)^2+\lambda^2}u_t$	$\sigma_x^2 = \frac{\lambda^2}{\alpha_c^2}\sigma_\pi^2$	$\left(1+\frac{\lambda(1-\rho)}{\alpha(1-\beta\rho)\rho\phi}\right)\pi_{t+1} + \frac{g_t}{\phi}$

6.4.3 The general solution under commitment

The globally optimal rule under commitment is likely not to fall within the restricted family of rules considered in the previous subsection. Remember that the restriction that we had so far was that the central bank was allowed to choose sequences for inflation and output gap that were a function only of *current* period realization of disturbances, so the solutions above were constrained optima within their family.

The constraint we have now is still:

$$
\pi_t = \lambda x_t + \beta E_t\pi_{t+1} + u_t
$$

Problem is:

$$
\max_{x_{t+1},\pi_{t+1}} -\frac{1}{2}E_t\left[\begin{array}{c}\alpha x_t^2 + \pi_t^2 + 2\xi_t\left(\pi_t - \lambda x_t - \beta\pi_{t+1} - u_t\right) + \\ \alpha\beta x_{t+1}^2 + \beta\pi_{t+1}^2 + 2\beta\xi_{t+1}\left(\pi_{t+1} - \lambda x_{t+1} - \beta\pi_{t+2} - u_{t+1}\right) + ...\end{array}\right]
$$

The First order condition's are, choosing x_{t+1} and π_{t+1} and scrolling them backward:

$$\alpha x_t = \xi_t \lambda$$
$$\xi_{t-1} = \pi_t + \xi_t$$

You will see the term ξ_{t-1} here. Under commitment (respectively, discretion), this term is non-zero (zero), since past actions are (are not) binding for the social planner. One can drop ξ_t to obtain:

$$\pi_t = p_t - p_{t-1} = -\frac{\alpha}{\lambda}(x_t - x_{t-1})$$
$$p_t = -\frac{\alpha}{\lambda}x_t$$

notice that this solution implies price level targeting (or if you like adjusting the change in the output gap in response to inflation). In fact another way to rewrite it is:

$$\log P_t = \log P^* - \frac{\alpha}{\lambda}x_t$$

where P^* is the target price level.

Definition: Price-level targeting (PT) is a policy systematically responds to deviations of the price level from the price level

target path to preclude long-run price-level drift.

This result can be compared with the optimal inflation-targeting policy that we obtained under discretion

$$\pi_t^b = -\frac{\alpha}{\lambda}x_t^b$$

6.4.3.1   Implementation of the general solution under commitment

A problem with such a rule (if policy responds to expected inflation and demand shocks) is that it might not guarantee real determinacy. This can be seen easily replacing the condition $\pi_{t+1} = (-\alpha/\lambda)(x_{t+1} - x_t)$ into the AD curve.

$$R_t^* = \frac{1}{\phi}(E_t x_{t+1} - x_t) + E_t \pi_{t+1} + \frac{1}{\phi}g_t =$$
$$= \left(1 - \frac{\lambda}{\phi\alpha}\right)E_t \pi_{t+1} + \frac{1}{\phi}g_t$$

A rule of this type might therefore permit self-fulfilling fluctuations in output and inflation that are clearly sub-optimal. Commitment to a rule of this kind might guarantee a REE only under some very stringent conditions.

6.4.4   Robustly optimal rules under commitment

Giannoni and Woodford (2002) solve a problem which is similar in nature to that above, however they explicitly allow for interest rate variability to enter the loss function of the central bank. That is:

$$\min W = E_0 \left[\sum_{t=0}^{\infty} \beta^t L_t \right]$$
$$L_t = \pi_t^2 + \mu_x x_t^2 + \mu_r r_t^2$$

The last term may arise if one takes into account the utility services of money, which were set arbitrarily close to zero in the derivation of the welfare function at the beginning of this section (see Woodford, Chapter 6.4.1)

Set up the Lagrangian:

$$
\begin{aligned}
L \;=\; & \pi_t^2 + \mu_x x_t{}^2 + \mu_r r_t^2 + \\
& \beta \pi_{t+1}^2 + \beta \mu_x x_{t+1}{}^2 + \beta \mu_r r_{t+1}^2 + \\
& -\xi_{1t}\left(\pi_t - \lambda x_t - \beta \pi_{t+1} - u_t\right) - \xi_{2t}\left(x_t - x_{t+1} + \phi R_t - \phi \pi_{t+1} - g_t\right) \\
& -\beta\left(\xi_{1t+1}\left(\pi_{t+1} - \lambda x_{t+1} - \beta \pi_{t+2} - u_{t+1}\right) - \beta \xi_{2t+1}\left(x_{t+1} - x_{t+2} + \phi R_{t+1} - \phi \pi_{t+2} - g_{t+1}\right)\right)
\end{aligned}
$$

The central bank minimizes over the whole time period, choosing π_{t+1} and x_{t+1} and r_{t+1}. Taking the first-order conditions and scrolling them one period backward yields:

$$
\begin{aligned}
\pi_t - \phi \beta^{-1} \xi_{1t-1} + \xi_{2t} - \xi_{2t-1} &= 0 \\
\mu_x x_t + \xi_{1t} - \beta^{-1} \xi_{1t-1} - k \xi_{2t} &= 0 \\
\mu_r r_t + \phi \xi_{1t} &= 0
\end{aligned}
$$

together with:

$$
\begin{aligned}
\pi_t - k x_t - \beta \pi_{t+1} &= 0 \\
x_t - E_t x_{t+1} + \phi r_t - \phi \pi_{t+1} - g_t &= 0
\end{aligned}
$$

This dynamic system of 5 equations in 5 unknowns can be solved for R as a function of existing endogenous variables only. Giannoni and Woodford call the deriving rule a "robustly optimal instrument rule". The idea is that you can play with the first three equations and solve for R_t as a function of the existing endogenous variables only.

$$
R_t = \left(1 + \frac{k\phi}{\beta}\right) R_{t-1} + \beta^{-1} \Delta R_{t-1} + \frac{k\phi}{\mu_r} \pi_t + \frac{\phi \mu_x}{\mu_r} \Delta x_t
$$

A bunch of comments:

- the optimal interest rate rule is a function only of the variables in the loss function

- the rule requires the interest rate to be positively related to fluctuations in current inflation, in changes of the output gap, and in lagged interest rates

- the rule is super-inertial, in the sense that it requires that the interest rate to vary by more than one for one to past fluctuations of the interest rate.

You will notice that, as μ_r approaches zero, we are back into the general solution under commitment result. In fact

$$
\begin{aligned}
k \pi_t &= \mu_x\left(x_{t-1} - x_t\right) \\
P_t &= -\frac{\mu_x}{k} x_t
\end{aligned}
$$

The outcome under timeless precommitment is shown in the Figure below (for λ_r arbitrarily low, file giannoni.m). Despite the fact that the cost shock has no persistence, the output gap displays positive serial correlation. By keeping output below potential for several periods into the future after the negative shock, the central bank is able to lower expectations of future inflation. A fall in $E_t \pi_{t+1}$ at the time of the shock improves the trade-off between inflation and output gap stabilization faced by the central bank.

This is unlike the case we obtain under discretion, where we find that

$$
\begin{aligned}
\pi_t^b &= \frac{\alpha}{\alpha\left(1 - \beta \rho\right) + \lambda^2} u_t \\
x_t^b &= -\frac{\lambda}{\alpha\left(1 - \beta \rho\right) + \lambda^2} u_t
\end{aligned}
$$

hence in that case the variables inherit the persistence properties of the cost-push shock, and there is no inertia in the variables following a shock.

OPTIMAL RESPONSES TO PURELY TRANSITORY SHOCKS, (TIMELESS) PRECOMMITMENT
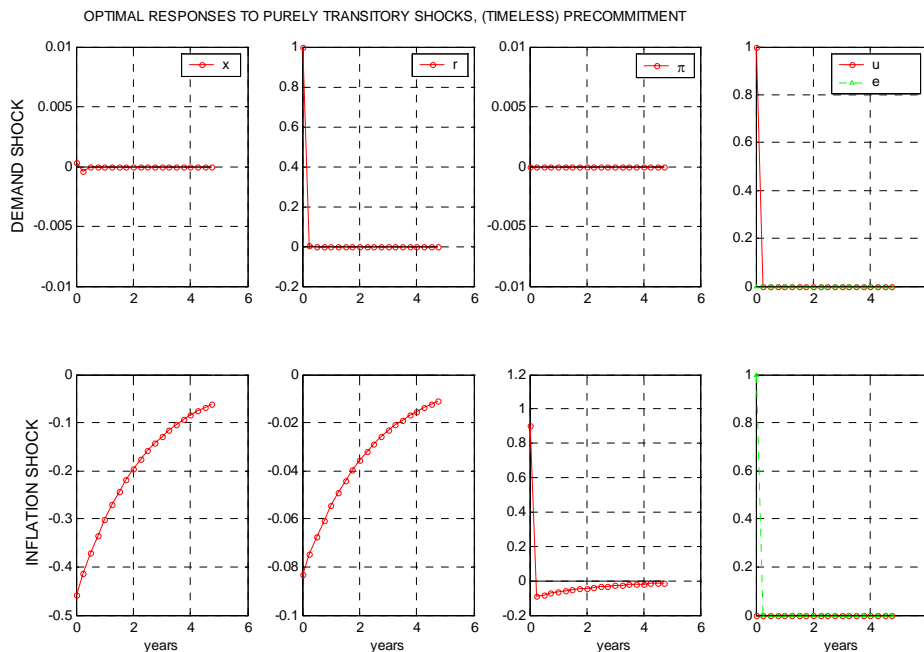
Responses to purely transitory shocks, timeless precommitment.

6.5 What about the Taylor rules?

Suppose we compare:

$$R_t = \left(1 + \frac{k\phi}{\beta}\right) R_{t-1} + \beta^{-1}\Delta R_{t-1} + \frac{k\phi}{\mu_r}\pi_t + \frac{\phi\mu_x}{\mu_r}\Delta x_t \tag{1}$$

$$R_t = \phi_x P_t + \phi_x x_t \tag{2}$$

$$R_t = \phi_x \pi_t + \phi_x x_t \tag{3}$$

which policy will yield higher welfare? To begin with, we know that 1 dominates 2 and 3 since it is, by construction, the optimal policy. What about 2 (Wicksellian rule, after Wicksell, 1907) versus 3 (Taylor)?

One would believe that 3 is better than 2, since under 2 the policymaker responds to an inflationary shock by bringing about deflation in future periods, hence lowering welfare. However, as shown for instance in Giannoni (2000), "Commitment to an optimal Wicksellian policy allows the policymaker to achieve a response of endogenous variables that is closer to the optimal plan than is the case with the optimal Taylor rule. One particularity of the equilibrium resulting from a Wicksellian policy is that the price level is stationary. This feature turns out to affect the response of endogenous variables in particular when shocks are very persistent [...] [T]he mere expectation of future deflation [..] under the optimal plan and the optimal non-inertial plan already depresses inflation when the shock hits the economy, and is expected to keep inflation below steady-state for several periods. In contrast, under optimal Wicksellian policy, both inflation and the price level rise strongly on impact, but they are expected to return progressively to their initial steady-state"

6.6 A digression on the supply shock

The source of the output-inflation variance trade-off for central bank is the "supply" shock. Were there only demand type shocks, there would be no trade-off. This is summarized in Erceg, Henderson and Levin (2000),

Proposition 2.

Proposition 2: With staggered price contracts and completely flexible wages, monetary policy can completely stabilize price inflation and the output gap, thereby attaining the Pareto-optimal social welfare level.

However, a price inflation / output gap variance trade-off arises endogenously in the model above with staggered wage and price setting. When both prices and wages are staggered, it is impossible for monetary policy to attain the Pareto optimum except in the special cases where either wages or prices are completely flexible. Nominal wage inflation and price inflation would remain constant only if the aggregate real wage rate were continuously at its Pareto-optimal level. Such an outcome is impossible because the Pareto-optimal real wage moves in response to various shocks, whereas the actual real wage could never change in the absence of nominal wage or price adjustment. Given that the Pareto optimum is infeasible, the monetary policymaker faces trade-offs in stabilizing wage inflation, price inflation, and the output gap.

(B) With staggered wage contracts and completely flexible prices, monetary policy can completely stabilize wage inflation and the output gap, thereby attaining the Pareto-optimal social welfare level.