The Solution of Linear Difference Systems under Rational Expectations

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8 October 2002

1. Linear difference systems under rational expectations: background. The discussion here follows King and Watson (1997), The solution of Linear Difference Systems under Rational Expectations, available at http://www.wws.princeton.edu/~mwatson/papers/algor6.pdf

Suppose we have a dynamic system described:

$$E_t y_{t+1} = W y_t + M E_t x_t$$

where y is the vector of endogenous variables and x is the vector of disturbance terms.

We partition y_t into non-predetermined (jump) variables p and predetermined k.

The Jordan form of $W = CJC^{-1}$ and $C^{-1}W = JC^{-1}$ allow us to write $\left(C^{-1} \equiv L\right)$[1]:

$$L E_t y_{t+1} = L W y_t + L M E_t x_t$$

i.e., defining $y^* = Ly$

$$E_t y^*_{t+1} = LWL^{-1} L y_t + LME_t x_t = LWL^{-1} y^*_t + LME_t x_t = J y^*_t + LME_t x_t$$

We then can write our earlier dynamic system as:

$$E_t \begin{bmatrix} u_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} u_t \\ s_t \end{bmatrix} + \begin{bmatrix} C_u^* \\ C_s^* \end{bmatrix} x_t$$

(where the mnemonic is that the u are the unstable canonical variables and the s are the stable canonical variables, i.e the partition $y^* = [u' \quad s']'$ is based on locating the unstable eigenvalues in J_u and the stable in J_s).

Since the eigenvalues of J_u are greater than one in modulus, the stability requirements is that the equations for u_t are solved forward as in Sargent and Wallace (1975). This yields:

$$u_t = -\sum_{h=0}^{\infty} J_u^{-(h+1)} C_u^* E_t x_{t+h}$$

Define the transformations:

$$\begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} L_{up} & L_{uk} \\ L_{sp} & L_{sk} \end{bmatrix} \begin{bmatrix} p \\ k \end{bmatrix}$$

[1] An example: given $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1 & 1/2 & 1/4 \end{bmatrix}$, whose eigenvalues are: $1, \frac{1}{2}, \frac{1}{4}$

The Jordan form of W is given by $W = CJC^{-1}$, that is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{3} & 2 & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{3}{10} \end{bmatrix}$$

and

$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} C_{up} & C_{uk} \\ C_{sp} & C_{sk} \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix}$$

hence

$$u_t = L_{up}p_t + L_{uk}k_t$$

and, provided that L_{up} is nonsingular,

$$p_t = L_{up}^{-1}u_t - L_{up}^{-1}L_{uk}k_t \tag{1}$$

Similarly, at each date t,
$k_t = C_{sp}u_t + C_{sk}s_t$ hence
Use then $E_t s_{t+1} = J_s s_t + C_s^* x_t$ and $s_t = L_{sp}p_t + L_{sk}k_t$ to get
$k_{t+1} = C_{sp}Eu_{t+1} + C_{sk}Es_{t+1} =$
$= C_{sp}L_{up}p_{t+1} + L_{uk}k_{t+1} + C_{sk}\left(J_s s_t + C_s^* x_t\right) =$
$= C_{sp}L_{up}\left(L_{up}^{-1}u_{t+1} - L_{up}^{-1}L_{uk}k_{t+1}\right) + L_{uk}k_{t+1} + C_{sk}\left(J_s s_t + C_s x_t\right) =$
$= C_{sp}u_{t+1} - C_{sp}L_{uk}k_{t+1} + L_{uk}k_{t+1} + C_{sk}J_s s_t + C_{sk}C_s x_t$
Thence
$k_{t+1} = C_{sp}u_{t+1} - C_{sp}L_{uk}k_{t+1} + L_{uk}k_{t+1} + C_{sk}J_s s_t + C_{sk}C_s x_t$
yielding:

$$k_{t+1} = \left(1 + C_{sp}L_{uk} - L_{uk}\right)^{-1}\left(C_{sp}u_{t+1} + C_{sk}J_s s_t + C_{sk}C_s x_t\right) \tag{2}$$

Equations (1) and (2) can be used recursively to construct $\{k_t, p_t, y_t\}_{t=1}^{\infty}$ given initial conditions for k_0 and x_0 together with the solution for u_1.

2. An application: the stochastic growth model with no depreciation. (SEE rbcprototype.m file on Matlab)

The representative individual solves:

$$\max E_t \left(\sum_{s=t}^{\infty} b^{s-t} \log C_s\right) \text{ s.t. } k_{t+1} - k_t = k_t^a - c_t$$

From loglinearization around the steady state of the first order conditions you get (see Obstfeld and Rogoff “Foundations of International Macroeconomics", page 502):

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} b^{-1} & \frac{b-1}{ab} \\ (1-a)(b-1) & 1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} + \begin{bmatrix} \frac{(1-a)(1-b)}{ab} \\ (1-a)(1-b) \end{bmatrix} [e_t]$$

Suppose $a = .5$, $b = .98$. Then:

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} .98^{-1} & \frac{.98-1}{.49} \\ (1-.5)(.98-1) & 1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} + \begin{bmatrix} \frac{.5(.02)}{.49} \\ (1-.5)(1-.98) \end{bmatrix} [e_t]$$

i.e:

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 1.0204 & -.0408 \\ -.01 & 1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} + \begin{bmatrix} .0204 \\ .01 \end{bmatrix} [e_t]$$

The Jordan form of $\begin{bmatrix} 1.0204 & -.0408 \\ -.01 & 1 \end{bmatrix}$ is CJC^{-1}:

$$\begin{bmatrix} \frac{51}{50} & \frac{-2}{50} \\ \frac{-1}{100} & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{10}\left(1+\sqrt{5}\right) & \frac{\sqrt{5}}{10}\left(\sqrt{5}-1\right) \\ -\frac{1}{10}\sqrt{5} & \frac{1}{10}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{101+\sqrt{5}}{100} & 0 \\ 0 & \frac{101-\sqrt{5}}{100} \end{bmatrix} \begin{bmatrix} 1 & 1-\sqrt{5} \\ 1 & 1+\sqrt{5} \end{bmatrix}^{-1}$$

Premultiply by C^{-1} the vector of variables, so that the new ones are $y^* = C^{-1}y$:

$$\begin{bmatrix} 1 & 1-\sqrt{5} \\ 1 & 1+\sqrt{5} \end{bmatrix} \begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} k_{t+1} - 1.2361c_{t+1} \\ k_{t+1} + 3.2361c_{t+1} \end{bmatrix}$$

The new system is therefore made up by two orthogonal combinations of c and k (with eigenvalues $\frac{101}{100} + \frac{\sqrt{5}}{100} = 1.032$ and $\frac{101}{100} - \frac{\sqrt{5}}{100} = .9877$):

$$\begin{bmatrix} k_{t+1} - 1.2361c_{t+1} \\ k_{t+1} + 3.2361c_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{101}{100} + \frac{\sqrt{5}}{100} & 0 \\ 0 & \frac{101}{100} - \frac{\sqrt{5}}{100} \end{bmatrix} \begin{bmatrix} k_t - 1.2361c_t \\ k_t + 3.2361c_t \end{bmatrix} + \begin{bmatrix} 8.0393 \times 10^{-3} \\ 5.2761 \times 10^{-2} \end{bmatrix} [e_t]$$

The explosive root $\left(\frac{101}{100} + \frac{\sqrt{5}}{100}\right)$ tells us that we need to solve the first equation forward, that is we need to impose:

$$k_{t+1} = 1.2361c_{t+1}, \text{ or } c_{t+1} = .809k_{t+1}$$

From the non-explosive part of the system (the line associated with the eigenvalue less than 1) we have:

$$k_{t+1} + 3.2361c_{t+1} = .98764\left(k_t + 3.2361c_t\right) + 0.053e_t$$

Replace the value of k_{t+1} with c_{t+1} and c_t with k_t in the non-explosive one:

$$1.236c_{t+1} + 3.236c_{t+1} = .9876\left(1.236c_t + 3.236c_t\right) + 0.053e_t$$

The solution is:

$$c_{t+1} = .79848k_t + 1.1852 \times 10^{-2}e_t \tag{1}$$

Go back to the equation for k_{t+1} and get:
$k_{t+1} = -3.2361\left(.79848k_t + 1.1852 \times 10^{-2}e_t\right) + .98764\left(k_t + 3.2361\left(.809k_t\right)\right) + 0.053e_t$.
The solution is

$$k_{t+1} = .98933k_t + 1.4646 \times 10^{-2}e_t \tag{2}$$

(1) and (2) constitute the linear decision rules of the system. They can be used to look at the impulse responses as a function of the e_t shock.