Technical Appendix for "Financial Business Cycles"

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This appendix contains some of the derivations for the paper "Financial Business Cycles".

1. The Extended Model

Here I present the extended model.

1.1. Household Savers

Savers solve

$$\max \sum_{t=0}^{\infty} \beta_H^t \left(A_{p,t} \left(1 - \eta\right) \log \left(C_{H,t} - \eta C_{H,t-1}\right) + j A_{j,t} A_{p,t} \log H_{H,t} + \tau \log \left(1 - N_{H,t}\right) \right)$$

subject to:

$$C_{H,t} + \frac{K_{H,t}}{A_{K,t}} + D_t + q_t \left(H_{H,t} - H_{H,t-1}\right) + ac_{KH,t} + ac_{DH,t}$$

$$= \left(R_{M,t} z_{KH,t} + \frac{1 - \delta_{KH,t}}{A_{K,t}} \right) K_{H,t-1} + R_{H,t-1} D_{t-1} + W_{H,t} N_{H,t} \tag{1.1}$$

where the adjustment costs take the following form

$$ac_{KH,t} = \frac{\phi_{KH}}{2} \frac{\left(K_{H,t} - K_{H,t-1}\right)^2}{K_H}$$

$$ac_{DH,t} = \frac{\phi_{DH}}{2} \frac{\left(D_t - D_{t-1}\right)^2}{D}$$

and the depreciation function is

$$\delta_{KH,t} = \delta_{KH} + b_{KH} \left(0.5 \zeta_H' z_{KH,t}^2 + \left(1 - \zeta_H'\right) z_{KH,t} + \left(0.5 \zeta_H' - 1\right) \right)$$

where $\zeta_H' = \frac{\zeta_H}{1 - \zeta_H}$ is a parameter measuring the curvature of the utilization rate function. $\zeta_H = 0$ implies $\zeta_H' = 0$; ζ_H approaching 1 implies ζ_H' approaches infinity and $\delta_{KH,t}$ stays constant. $b_{KH} = \frac{1}{\beta_H} + 1 - \delta_{KH}$ and implies a unitary steady state utilization rate. ac measures a quadratic adjustment cost for changing the quantity i between time t and time $t+1$. The adjustment cost is external. Habits are external too.

The household problem yields, denoting with $u_{CH,t} = \frac{A_{p,t}}{C_{H,t} - \eta C_{H,t-1}}$ and $u_{HH,t} = \frac{j A_{j,t} A_{p,t}}{H_{H,t}}$ the marginal

utilities of consumption and housing.

$$u_{CH,t}\left(1 + \frac{\partial ac_{DH,t}}{\partial D_t}\right) = \beta_H R_{H,t} u_{CH,t+1} \qquad (1.2)$$

$$W_{H,t} u_{CH,t} = \frac{\tau_H}{1 - N_{H,t}} \qquad (1.3)$$

$$\frac{1}{A_{K,t}} u_{CH,t}\left(1 + \frac{\partial ac_{KH,t}}{\partial K_{H,t}}\right) = \beta_H \left(R_{M,t+1} z_{KH,t+1} + \frac{1 - \delta_{KH,t+1}}{A_{K,t+1}}\right) u_{CH,t+1} \qquad (1.4)$$

$$q_t u_{CH,t} = u_{HH,t} + \beta_H q_{t+1} u_{CH,t+1} \qquad (1.5)$$

$$R_{M,t} = \delta'\left(z_{KH,t}\right) \qquad (1.6)$$

where $A_{K,t}$ is an investment shock, $A_{p,t}$ is a consumption preference shock, $A_{j,t}$ is a housing demand shock.

1.2. Household Borrowers

They solve

$$\max \sum_{t=0}^{\infty} \beta_S^t \left(A_{p,t}\left(1 - \eta\right)\log\left(C_{S,t} - \eta C_{S,t-1}\right) + j A_{j,t} A_{p,t} \log H_{S,t} + \tau \log\left(1 - N_{S,t}\right)\right)$$

subject to

$$C_{S,t} + q_t\left(H_{S,t} - H_{S,t-1}\right) + R_{S,t-1} L_{S,t-1} - \varepsilon_{H,t} + ac_{SS,t} = L_{S,t} + W_{S,t} N_{S,t} \qquad (1.7)$$

and to

$$L_{S,t} \leq \rho_S L_{S,t-1} + \left(1 - \rho_S\right) m_S A_{MH,t} \frac{q_{t+1}}{R_{S,t}} H_{S,t} - \varepsilon_{H,t} \qquad (1.8)$$

where $\varepsilon_{H,t}$ is the borrower repayment shock, $A_{M,t}$ is a loan-to-value ratio shock. The adjustment cost is

$$ac_{SS,t} = \frac{\phi_{SS}}{2} \frac{\left(L_{S,t} - L_{S,t-1}\right)^2}{L_S}$$

The first order conditions are, denoting with $u_{CS,t} = \frac{A_{p,t}}{C_{S,t}}$ and $u_{HS,t} = \frac{j A_{j,t} A_{p,t}}{H_{S,t}}$ the marginal utilities of consumption and housing; and with $\lambda_{S,t} u_{CS,t}$ the (normalized) multiplier on the borrowing constraint:

$$\left(1 - \frac{\partial ac_{SS,t}}{\partial L_{S,t}} - \lambda_{S,t}\right) u_{CS,t} = \beta_S\left(R_{S,t} - \rho_S \lambda_{S,t+1}\right) u_{CS,t+1} \qquad (1.9)$$

$$W_{S,t} u_{CS,t} = \frac{\tau_S}{1 - N_{S,t}} \qquad (1.10)$$

$$\left(q_t - \lambda_{S,t}\left(1 - \rho_S\right) m_S A_{MH,t} \frac{q_{t+1}}{R_{S,t}}\right) u_{CS,t} = u_{HS,t} + \beta_S q_{t+1} u_{CS,t+1} \qquad (1.11)$$

1.3. Bankers

Bankers solve

$$\max \sum_{t=0}^{\infty} \beta_B^t \log\left(C_{B,t} - \eta C_{B,t-1}\right)$$

subject to:

$$C_{B,t} + R_{H,t-1} D_{t-1} + L_{E,t} + L_{S,t} + ac_{DB,t} + ac_{EB,t} + ac_{SB,t} = D_t + R_{E,t} L_{E,t-1} + R_{S,t} L_{S,t-1} - \varepsilon_{E,t} - \varepsilon_{S,t} \quad (1.12)$$

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where $\varepsilon_{E,t}$ is the entrepreneur repayment shock. The adjustment costs are

$$ac_{DB,t} = \frac{\phi_{DB}}{2} \frac{(D_t - D_{t-1})^2}{D}$$

$$ac_{EB,t} = \frac{\phi_{EB}}{2} \frac{(L_{E,t} - L_{E,t-1})^2}{L_E}$$

$$ac_{SB,t} = \frac{\phi_{SB}}{2} \frac{(L_{S,t} - L_{S,t-1})^2}{L_S}$$

Denote $\varepsilon_t = \varepsilon_{E,t} + \varepsilon_{S,t}$. Let $L_t = L_{E,t} + L_{S,t}$. The banker's constraint is a capital adequacy constraint of the form:

$$\underbrace{(L_t - D_t - \varepsilon_t)}_{\text{bank equity}} \geq \rho_D (L_{t-1} - D_{t-1} - \varepsilon_{t-1}) + (1 - \gamma)(1 - \rho_D) \underbrace{(L_t - \varepsilon_t)}_{\text{bank assets}}$$

stating that bank equity (after losses) must exceed a fraction of bank assets, allowing for a partial adjustment in bank capital given by ρ_D. Such constraint can be rewritten as a leverage constraint of the form

$$D_t \leq \rho_D (D_{t-1} - (L_{E,t-1} + L_{S,t-1} - (\varepsilon_{E,t-1} + \varepsilon_{S,t-1}))) +$$
$$(1 - (1 - \gamma)(1 - \rho_D))(L_{Et} + L_{S,t} - (\varepsilon_{E,t} + \varepsilon_{S,t})) \tag{1.13}$$

The first order conditions to the banker's problem imply, choosing D, L_E, L_S and letting $\lambda_{B,t} u_{CB,t}$ be the normalized multiplier on the borrowing constraint:

$$\left(1 - \lambda_{B,t} - \frac{\partial ac_{DB,t}}{\partial D_t}\right) u_{CB,t} = \beta_B (R_{H,t} - \rho_D \lambda_{B,t+1}) u_{CB,t+1} \tag{1.14}$$

$$\left(1 - (\gamma_E (1 - \rho_D) + \rho_D) \lambda_{B,t} + \frac{\partial ac_{EB,t}}{\partial L_{E,t}}\right) u_{CB,t} = \beta_B (R_{E,t+1} - \rho_D \lambda_{B,t+1}) u_{CB,t+1} \tag{1.15}$$

$$\left(1 - (\gamma_S (1 - \rho_D) + \rho_D) \lambda_{B,t} + \frac{\partial ac_{SB,t}}{\partial L_{S,t}}\right) u_{CB,t} = \beta_B (R_{S,t} - \rho_D \lambda_{B,t+1}) u_{CB,t+1} \tag{1.16}$$

1.4. Entrepreneurs

Entrepreneurs obtain loans and produce goods (including capital). Entrepreneurs hire workers and demand capital supplied by the household sector.

$$\max \sum_{t=0}^{\infty} \beta_E^t \log (C_{E,t} - \eta C_{E,t-1})$$

subject to:

$$C_{E,t} + \frac{K_{E,t}}{A_{K,t}} + q_t H_{E,t} + R_{E,t} L_{E,t-1} + W_{H,t} N_{H,t} + W_{S,t} N_{S,t} + R_{M,t} z_{KH,t} K_{H,t-1} \tag{1.17}$$

$$= Y_t + \frac{1 - \delta_{KE,t}}{A_{K,t}} K_{E,t-1} + q_t H_{E,t-1} + L_{E,t} + \varepsilon_{E,t} + ac_{KE,t} + ac_{EE,t}$$

and to

$$Y_t = A_{Z,t} (z_{KH,t} K_{H,t-1})^{\alpha\mu} (z_{KE,t} K_{E,t-1})^{\alpha(1-\mu)} H_{E,t-1}^{\nu} N_{H,t}^{(1-\alpha-\nu)(1-\sigma)} N_{S,t}^{(1-\alpha-\nu)\sigma} \tag{1.18}$$

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where $A_{Z,t}$ is a shock to total factor productivity. The adjustment costs are

$$ac_{KE,t} = \frac{\phi_{KE}}{2}\frac{(K_{E,t} - K_{E,t-1})^2}{K_E}$$

$$ac_{EE,t} = \frac{\phi_{EE}}{2}\frac{(L_{E,t} - L_{E,t-1})^2}{L_E}$$

Note that symmetrically to the household problem entrepreneurs are subject to an investment shock, can adjust the capital utilization rate, and pay a quadratic capital adjustment cost. The depreciation rate is governed by

$$\delta_{KE,t} = \delta_{KE} + b_{KE}\left(0.5\zeta'_E z^2_{KE,t} + \left(1 - \zeta'_E\right) z_{KE,t} + \left(0.5\zeta'_E - 1\right)\right)$$

where setting $b_{KE} = \frac{1}{\beta_E} + 1 - \delta_{KE}$ implies a unitary steady state utilization rate.

Entrepreneurs are subject to a borrowing/pay in advance constraint that acts as a wedge on the capital and labor demand. The constraint is

$$L_{E,t} = \rho_E L_{E,t-1} + (1 - \rho_E) A_{ME,t}\left(m_H\frac{q_{t+1}}{R_{E,t+1}}H_{E,t} + m_K K_{E,t} - m_N\left(W_{H,t}N_{H,t} + W_{S,t}N_{S,t}\right)\right) \quad (1.19)$$

Letting $u_{CE,t}$ be the marginal utility of consumption and $\lambda_{E,t}u_{CE,t}$ the normalized borrowing constraint, the first order conditons for L_E, K_E and H_E are:

$$\left(1 - \lambda_{E,t} + \frac{\partial ac_{LE,t}}{\partial L_{E,t}}\right)u_{CE,t} = \beta_E\left(R_{E,t+1} - \rho_E\lambda_{E,t+1}\right)u_{CE,t+1} \quad (1.20)$$

$$\left(1 + \frac{\partial ac_{KE,t}}{\partial K_{E,t}} - \lambda_{E,t}\left(1 - \rho_E\right)m_K A_{ME,t}\right)u_{CE,t} = \beta_E\left(1 - \delta_{KE,t+1} + R_{K,t+1}z_{KE,t+1}\right)u_{CE,t+1} \quad (1.21)$$

$$\left(q_t - \lambda_{E,t}\left(1 - \rho_E\right)m_H A_{ME,t}\frac{q_{t+1}}{R_{E,t+1}}\right)u_{CE,t} = \beta_E q_{t+1}\left(1 + R_{V,t+1}\right)u_{CE,t+1} \quad (1.22)$$

Additionally, these conditions can be combined with those of the production arm of the firm, giving

$$\alpha\mu Y_t = R_{K,t}z_{KE,t}K_{E,t-1} \quad (1.23)$$
$$\alpha\left(1 - \mu\right)Y_t = R_{M,t}z_{KH,t}K_{H,t-1} \quad (1.24)$$
$$\nu Y_t = R_{V,t}q_t H_{E,t-1} \quad (1.25)$$
$$\left(1 - \alpha - \nu\right)\left(1 - \sigma\right)Y_t = W_{H,t}N_{H,t}\left(1 + m_N A_{ME,t}\lambda_{E,t}\right) \quad (1.26)$$
$$\left(1 - \alpha - \nu\right)\sigma Y_t = W_{S,t}N_{S,t}\left(1 + m_N A_{ME,t}\lambda_{E,t}\right) \quad (1.27)$$
$$R_{K,t} = \delta'\left(z_{KE,t}\right) \quad (1.28)$$

1.5. Equilibrium

Market clearing is implied by Walras's law by aggregating all the budget constraints. For housing, we have the following market clearing condition

$$H_{H,t} + H_{S,t} + H_{E,t} = 1 \quad (1.29)$$

The model endogenous variables are
14: quantities Y H_E H_H H_S K_E K_H N_H N_S C_B C_E C_H C_S z_{KH} z_{KE}
3: loans & deposits L_E L_S D
3: prices q W_H W_S
6: interest rates R_K R_M R_V R_E R_S R_H
3: multipliers λ_E λ_S λ_B

together with the definition of the depreciation rate functions and the adjustment cost functions given in the text above.

1.6. Shocks

The following zero-mean, AR(1) shocks are present in the estimated version of the model

$$\varepsilon_{E,t} \quad \varepsilon_{H,t} \quad \log A_{j,t} \quad \log A_{K,t} \quad \log A_{ME,t} \quad \log A_{MH,t} \quad \log A_{p,t} \quad \log A_{z,t}$$

The shocks follow the processes given by:

$$
\begin{aligned}
\varepsilon_{E,t} &= \rho_{be}\varepsilon_{E,t-1} + u_{E,t}, \quad u_E \sim N(0, \sigma_{be}) \\
\varepsilon_{H,t} &= \rho_{bh}\varepsilon_{H,t-1} + \upsilon_{H,t}, \quad u_H \sim N(0, \sigma_{bh}) \\
\log A_{j,t} &= \rho_j \log A_{j,t-1} + \upsilon_{j,t}, \quad u_j \sim N(0, \sigma_j) \\
\log A_{K,t} &= \rho_K \log A_{K,t-1} + \upsilon_{K,t}, \quad u_K \sim N(0, \sigma_k) \\
\log A_{ME,t} &= \rho_{me} \log A_{ME,t-1} + \upsilon_{ME,t}, \quad u_{ME} \sim N(0, \sigma_{me}) \\
\log A_{MH,t} &= \rho_{mh} \log A_{MH,t-1} + \upsilon_{MH,t}, \quad u_{MH} \sim N(0, \sigma_{mh}) \\
\log A_{p,t} &= \rho_p \log A_{p,t-1} + \upsilon_{p,t}, \quad u_p \sim N(0, \sigma_p) \\
\log A_{Z,t} &= \rho_z \log A_{Z,t-1} + \upsilon_{z,t}, \quad u_z \sim N(0, \sigma_z)
\end{aligned}
$$

2. Steady State of Extended Model

Interest Rates and Multipliers. Let's begin with them

$$
\begin{align}
R_H &= 1/\beta_H \tag{2.1}\\
\lambda_B &= \frac{1 - \beta_B R_H}{1 - \beta_B \rho_D} \tag{2.2}\\
R_E &= \frac{1}{\beta_B} - \frac{(1 - \beta_B)\rho_D + (1 - \rho_D)\gamma_E}{\beta_B}\frac{1 - \beta_B R_H}{1 - \beta_B \rho_D} \tag{2.3}\\
R_S &= \frac{1}{\beta_B} - \frac{(1 - \beta_B)\rho_D + (1 - \rho_D)\gamma_S}{\beta_B}\frac{1 - \beta_B R_H}{1 - \beta_B \rho_D} \tag{2.4}\\
\lambda_E &= \frac{1 - \beta_E R_E}{1 - \beta_E \rho_E} \tag{2.5}\\
\lambda_S &= \frac{1 - \beta_S R_S}{1 - \beta_S \rho_S} \tag{2.6}\\
R_V &= 1/\beta_E - 1 - \lambda_E \frac{(1 - \rho_E)m_H}{\beta_E R_E} \tag{2.7}\\
R_K &= 1/\beta_E - (1 - \delta) - \lambda_E m_K/(1 - \lambda_E) \tag{2.8}\\
R_M &= 1/\beta_H - (1 - \delta) \tag{2.9}
\end{align}
$$

This implies the following restrictions on discount factors (for multipliers to be positive).

$$\lambda_B > 0 \text{ if } \beta_B < \beta_H$$

$$\lambda_E > 0 \text{ if } \beta_E R_E < 1$$

$$\beta_E\left(1 - ((1 - \beta_B)\rho_D + (1 - \rho_D)\gamma_E)\frac{1 - \beta_B R_H}{1 - \beta_B \rho_D}\right) < \beta_B$$

$$\text{if } \rho = 0$$

$$\beta_E < \frac{1}{\frac{1 - \gamma_E}{\beta_B} + \frac{\gamma_E}{\beta_H}}$$

$$\lambda_S > 0 \text{ if } \beta_S R_S < 1$$

$$\beta_S\left(1 - ((1 - \beta_B)\rho_D + (1 - \rho_D)\gamma_S)\frac{1 - \beta_B R_H}{1 - \beta_B \rho_D}\right) < \beta_B$$

Some Useful Constants. List of constants

$$oo_1 = \frac{j}{1 - \beta_H} \tag{2.10}$$

$$oo_2 = \frac{j}{1 - \beta_S - \lambda_S \left(1 - \rho_S\right) m_S / R_S} \tag{2.11}$$

$$oo_3 = \frac{1}{1 + \left(1 - 1/R_S\right) m_S oo_2} \tag{2.12}$$

$$oo_4 = \gamma_E \left(R_H - 1\right) \left(\frac{\nu m_H}{R_V R_E} + \frac{\mu \alpha m_K}{R_K R_E} - \frac{1 - \alpha - \nu}{1 + m_N \lambda_E} m_N \right) \tag{2.13}$$

$$oo_5 = \gamma_S \frac{m_S}{R_S} oo_2 oo_3 \left(R_H - 1\right) \tag{2.14}$$

$$oo_6 = R_M - \delta \tag{2.15}$$

$$oo_7 = \frac{\left(1 - \alpha - \nu\right)\left(1 - \sigma\right)}{1 + m_N \lambda_E} \tag{2.16}$$

$$oo_8 = \left(1 - \mu\right) \frac{\alpha}{R_M} \tag{2.17}$$

Start with Ratios, then Move to Levels. The housing consumption ratios are:

$$qH_H = oo_1 C_H \tag{2.18}$$

$$qH_S = oo_2 C_S \tag{2.19}$$

$$qH_E = \frac{\nu}{R_V} Y \tag{2.20}$$

The Households

$$C_H = \left(R_M - \delta\right) K_H + \left(R_H - 1\right) D + W_H N_H \tag{2.21}$$

Use

$$D = \gamma_E L_E + \gamma_S L_S = \gamma_E \left(m_H q H_E / R_E + m_K K_E / R_E - m_N \left(W_H N_H + W_S N_S\right)\right) + \gamma_S m_S q H_S / R_S \tag{2.22}$$

so

$$C_H = \left(R_M - \delta\right) K_H + \gamma_E \left(R_H - 1\right) \left(\frac{m_H}{R_E} q H_E + \frac{m_K}{R_R} K_E - m_N \left(W_H N_H + W_S N_S\right)\right) \tag{2.23}$$
$$+ \gamma_S \frac{m_S}{R_S} \left(R_H - 1\right) q H_S + W_H N_H + F$$

Also

$$C_S = W_S N_S - \left(1 - 1/R_S\right) m_S q H_S \tag{2.24}$$

Hence C_S and C_H can be rewritten as

$$C_S = \frac{1}{1 + \left(1 - 1/R_S\right) m_S oo_2} W_S N_S = oo_3 W_S N_S \tag{2.25}$$

$$C_H = \left(R_M - \delta\right) K_H + \gamma_E \left(R_H - 1\right) \left(\frac{m_H}{R_E} q H_E + \frac{m_K}{R_R} K_E - m_N \left(W_H N_H + W_S N_S\right)\right)$$
$$+ \gamma_S \frac{m_S}{R_S} \left(R_H - 1\right) oo_2 oo_3 W_S N_S + W_H N_H \tag{2.26}$$
$$= oo_6 K_H + oo_4 Y + oo_5 W_S N_S + W_H N_H \tag{2.27}$$

Use
$$W_H N_H = oo_7 Y \tag{2.28}$$

so that
$$C_H = oo_6 K_H + \left(1 + \frac{oo_4}{oo_7}\right) W_H N_H + oo_5 W_S N_S \tag{2.29}$$

Labor Market Equilibrium.   Labor supplies

$$\frac{W_H}{C_H} = \frac{\tau_H}{1 - N_H} \tag{2.30}$$

$$\frac{W_S}{C_S} = \frac{\tau_S}{1 - N_S} \tag{2.31}$$

We want to solve this mess for N. The trick is to write everything as a function of $W_H N_H$ in the budget constraint. Note that

$$W_H N_H = oo_7 Y \tag{2.32}$$

$$W_S N_S = \frac{\sigma}{1-\sigma} W_H N_H \tag{2.33}$$

$$K_H = oo_8 Y \tag{2.34}$$

$$K_H = \frac{oo_8}{oo_7} W_H N_H \tag{2.35}$$

Hence:

$$\frac{W_H}{\left(oo_6 \frac{oo_8}{oo_7} + 1 + \frac{oo_4}{oo_7} + oo_5 \frac{\sigma}{1-\sigma}\right) W_H N_H} = \frac{\tau_H}{1 - N_H} \tag{2.36}$$

$$\frac{W_S}{oo_3 W_S N_S} = \frac{\tau_S}{1 - N_S} \tag{2.37}$$

Hence from all of this we get, using

$$z_1 = oo_6 \frac{oo_8}{oo_7} + 1 + \frac{oo_4}{oo_7} + oo_5 \frac{\sigma}{1-\sigma} \tag{2.38}$$

$$\frac{1}{z_1 N_H} = \frac{\tau_H}{1 - N_H} \; --> \; N_H = \frac{1}{1 + \tau_H z_1} \tag{2.39}$$

and likewise
$$N_S = \frac{1}{1 + oo_3 \tau_S} \tag{2.40}$$

The Consumption Output Ratios.   Given the housing consumption ratios above, using the following consumption output ratios

$$cy_S = oo_3 \frac{(1 - \alpha - \nu)\sigma}{1 + \lambda_E} \tag{2.41}$$

$$cy_H = z_1 oo_7 \tag{2.42}$$

$$cy_E = \alpha\mu + \nu - \gamma_E (R_H - 1)\left(\frac{m_H}{R_E} q H_E + \frac{m_K}{R_R} K_E - m_N (W_H N_H + W_S N_S)\right) - \delta\frac{\alpha\mu}{R_K} \tag{2.43}$$

one can solve for H can be solved using the stuff below. Note that the cy_E ratio is not needed, I just

put it there in case.

$$qH_H = oo_1 cy_H Y \tag{2.44}$$
$$qH_S = oo_2 cy_S Y \tag{2.45}$$
$$qH_E = \frac{\nu}{R_V} Y \tag{2.46}$$
$$H_H + H_S + H_E = 1 \tag{2.47}$$

Finally, the Levels.   Hence

$$Y = \left(\frac{\alpha(1-\mu)}{R_M}\right)^{\frac{\alpha(1-\mu)}{1-\alpha}} \left(\frac{\alpha\mu}{R_K}\right)^{\frac{\alpha\mu}{1-\alpha}} H_E^{\frac{\nu}{1-\alpha}} \left(N_H^{1-\sigma} N_S^{\sigma}\right)^{\frac{1-\alpha-\nu}{1-\alpha}} \tag{2.48}$$

$$K_E = \mu\alpha\frac{Y}{R_K} \tag{2.49}$$

$$K_H = (1-\mu)\alpha\frac{Y}{R_M} \tag{2.50}$$

$$L_E = m_H q\frac{H_E}{R_E} + m_K \frac{K_E}{R_E} - m_N \frac{1-\alpha-\nu}{1+m_N\lambda_E} Y \tag{2.51}$$

$$W_H = \frac{1-\alpha-\nu}{1+m_N\lambda_E} \frac{(1-\sigma)Y}{N_H} \tag{2.52}$$

$$W_S = \frac{1-\alpha-\nu}{1+m_N\lambda_E} \frac{\sigma Y}{N_S} \tag{2.53}$$

$$L_S = \frac{m_S}{R_S} oo_2 C_S = \frac{m_S}{R_S} oo_2 oo_3 W_S N_S \tag{2.54}$$

Then solve for C_B, C_E and C_S as well as D (can use simpler formulas now). I would use ratios defined above for cy_S, cy_H and cy_E and then go back to levels for bankers

$$D = \gamma_E L_E + \gamma_S L_S \tag{2.55}$$
$$C_B = (R_E - 1)L_E + (R_S - 1)L_S - (R_H - 1)D \tag{2.56}$$

3. Derivation of Demand and Supply Curves in Figure 5

In this framework, one can plot and analyze demand and supply curves very easily. Let the adjustment costs functions and derivatives be written in this form

$$\Phi_t = \frac{\phi_{LE}}{2} \frac{(L_t - L_{t-1})^2}{L_E}$$

$$\phi_t = \frac{d\Phi_t}{dL_t} = \frac{\phi_{LE}}{L} (L_t - L_{t-1}) = \phi' (L_t - L_{t-1})$$

Besides, assume that adjustment costs are external, so that the expectation of future adjustment costs does not show up in the Euler equation.

Using this functional form, the parameter ϕ_{LE} ends up measuring the semielasticity of loan demand to the interest rates (up to first order, as will be shown below). To be more precise, when quarterly interest rates rise by 25 basis points say (annual rates go up by 100 basis points), loan demand falls in percentage terms by $0.25/\phi_{LE}$. A $\phi_{LE} = 0.25$ implies for instance a drop in loan demand by 1 percent following a 1% rise in interest rates.

3.1. Entrepreneurs: Loan demand

To save on notation, I let $L_{E,t}$ be L_t in what follows.

Loan demand is the schedule described by equation 1.20. Taking into account adjustment costs, the loan demand equation is implicitly defined by the following equation, where L_t and R_{t+1} are loan demand and loan rate, and everything else is a shifter.

$$\left(1 - \frac{\phi_{LE}}{L} (L_t - L_{t-1}) - \lambda_{E,t}\right) C_{E,t}^{-\sigma_E} = \beta_E (R_{E,t+1} - \rho\lambda_{E,t+1}) C_{E,t+1}^{-\sigma_E} \tag{1\_d}$$

When a shock hits that affects all variables, one can trace how loan demand reacts but plotting all combinations of L and R such that the equation above holds. I just linearize this equation around the steady state to make it more transparent. After linearization, this equation becomes:

$$L_t = L_{t-1} - \frac{1}{\phi'_{LE}} (\lambda_{E,t} - \lambda_E) - \frac{\sigma_E (1 - \lambda_E)}{\phi'_{LE}} (C_{E,t} - C_{E,t+1}) - \frac{1}{\phi'_{LE}} \frac{1 - \lambda_E}{R_E - \rho_E \lambda_E} (R_{E,t+1} - R_E - \rho_E (\lambda_{E,t+1} - \lambda_E))$$

$$\tag{1\_d\_linearized}$$

where a variable without time subscript denotes the steady–state value.

Using this equation, one can note that, when ρ_E is zero, for small values of λ_E and R_E close to one, we have that

$$\frac{dL_t/L}{dR_{t+1}} = \frac{1}{\phi_{LE}}$$

hence $\frac{1}{4\phi_{LE}}$ is the semielasticity of loan demand to changes in the annualized interest rate.

3.2. Banks: Loan Supply

Loan supply is given by equation 1.15

$$\left(1 + \phi'_{LB} (L_t - L_{t-1}) - (\gamma_E (1 - \rho_D) + \rho_D) \lambda_{B,t}\right) C_{B,t}^{-\sigma_B} = \beta_B \left(\frac{R_{E,t+1}}{\pi_{t+1}} - \rho_D\lambda_{B,t+1}\right) C_{B,t+1}^{-\sigma_B} \tag{1\_s}$$

After we rearrange, denoting with $\chi_B = \gamma_E (1 - \rho_D) + \rho_D$, we have

$$L_t = L_{t-1} + \frac{\chi_B}{\phi'_{LB}} (\lambda_t - \lambda_B) + \frac{\sigma_B (1 - \chi_B\lambda)}{\phi'_{LB} C_B} (C_t - C_{t+1}) + \frac{1}{\phi'_{LB}} \frac{1 - \chi_B\lambda_B}{R_E - \rho_D\lambda_B} (R_{t+1} - R_E - \rho_D (\lambda_{B,t+1} - \lambda_B))$$

$$\tag{1\_s\_linearized}$$

For small ρ_D, $1/(4\phi_{LB})$ is the elasticity of loan supply to changes in the interest rate.

3.3. Households: Deposits Supply

Deposit supply is the linearized version of equation 1.2. Starting from

$$u_{CH,t}\left(1 + \frac{\partial ac_{DH,t}}{\partial D_t}\right) = \beta_H \frac{R_{H,t}}{\pi_{t+1}} u_{CH,t+1} \tag{d\_s}$$

we can rewrite this as

$$\begin{aligned}
\log u_{CH,t} + \log\left(1 + \frac{\partial ac_{DH,t}}{\partial D_t}\right) &= \log\beta + \log R_t + \log u_{CH,t+1} \\
\widehat{u}_{CH,t} + \phi\left(\widehat{D}_t - \widehat{D}_{t-1}\right) &= \widehat{R}_t + \widehat{u}_{CH,t+1} \\
\widehat{D}_t - \widehat{D}_{t-1} &= \frac{1}{\phi}\left(\widehat{R}_t + \widehat{u}_{CH,t+1} - \widehat{u}_{CH,t}\right)
\end{aligned}$$

Therefore this simplifies to (in levels)

$$D_t - D_{t-1} = \frac{D_{SS}}{\phi}\left(\frac{R_t - R_{SS}}{R_{SS}} + \frac{u_{CH,t+1} - u_{CH,t}}{u_{CH,SS}}\right) \tag{d\_s\_linearized}$$

3.4. Banks: Deposit Demand

Deposit demand is given by equation 1.14

$$\left(1 - \phi'_{DB}\left(D_t - D_{t-1}\right) - \lambda_{B,t}\right) C_{B,t}^{-\sigma_B} = \beta_B\left(\frac{R_{H,t}}{\pi_{t+1}} - \rho_D \lambda_{B,t+1}\right) C_{B,t+1}^{-\sigma_B} \tag{d\_d}$$

After linearization, this equation becomes

$$D_t = D_{t-1} - \frac{1}{\phi'_{DB}}\left(\lambda_{B,t} - \lambda_B\right) - \frac{\sigma_B\left(1 - \lambda_B\right)}{\phi'_{DB} C_B}\left(C_{B,t} - C_{B,t+1}\right) - \frac{1}{\phi'_{DB}}\frac{1 - \lambda_B}{R_H - \rho_D\lambda_B}\left(R_{H,t} - R_H - \rho_D\left(\lambda_{B,t+1} - \lambda_B\right)\right)$$
$$\tag{d\_d\_linearized}$$

3.5. Entrepreneurs: Capital Supply

Capital supply is given by equation 1.21:

$$\left(1 + \phi'_{KE}\left(K_{E,t} - K_{E,t-1}\right) - \left(1 - \rho_E\right) m_K \lambda_{E,t}\right) C_{E,t}^{-\sigma_E} = \beta_E\left(1 - \delta_{KE,t+1} + R_{K,t+1}\right) C_{E,t+1}^{-\sigma_E} \tag{k\_s}$$

After we arrange, letting $\chi_E = \left(\left(1 - \rho_E\right) m_K\right)$

$$K_{E,t} = K_{E,t-1} + \frac{\chi_E}{\phi'_{KE}}\left(\lambda_{E,t} - \lambda_E\right) + \frac{\sigma_E\left(1 - \chi_E\lambda_E\right)}{\phi'_{KE} C_E}\left(C_{E,t} - C_{E,t+1}\right)$$
$$+ \frac{1}{\phi'_{KE}}\frac{1 - \chi_E\lambda_E}{R_{KE} + 1 - \delta_{KE}}\left(R_{KE,t+1} - R_{KE} - \left(\delta_{KE,t+1} - \delta_{KE}\right)\right) \tag{k\_s\_linearized}$$

If adding LTV shocks, $K_{E,t} = K_{E,t-1} + \frac{\chi_E}{\phi'_{KE}}\left(\lambda_{E,t} - \lambda_E\right) + \frac{\lambda_E\chi_E/m_K}{\phi'_{KE}}\left(m_{K,t} - m_K\right) + ...$
Since

$$\begin{aligned}
R_{K,t} &= b_{KE}\left(\zeta'_E z_{KE,t} + 1 - \zeta'_E\right) z_{KE,t} \\
\delta_{KE,t} &= \delta_{KE} + b_{KE}\left(0.5\zeta' z_{KE,t}^2 + \left(1 - \zeta'\right) z_{KE,t} + \left(0.5\zeta' - 1\right)\right)
\end{aligned}$$

It makes sense to express capital supply as a function of δ which depends on R. To do so note that

$$
\begin{aligned}
\delta_t - \delta &= b_{KE} \left(z_t - z \right) \\
R_{K,t} - R &= b_{KE} \left(1 + \zeta' \right) \left(z_t - z \right) \\
\delta_t - \delta &= \frac{1}{1 + \zeta'} \left(R_{K,t} - R \right) = \left(1 - \zeta \right) \left(R_{K,t} - R \right)
\end{aligned}
$$

which highlights how ζ makes capital supply more or less elastic.

3.6. Firms: Capital Demand

Capital demand is the remarkably linearized version of equation 1.23 (using the formula for Y_t given by the production function) that this page is too small to contain.