Appendix E: Mathematical Derivations for the equations of "Housing Market Spillovers: Evidence from an Estimated DSGE Model"

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March 3, 2009

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Appendix E: Mathematical Derivations for the Equations of "Housing Market Spillovers: Evidence from an Estimated DSGE Model"

1 The model

1.1 Patient households

Lifetime utility is given by:

$$V_t = E_0 \sum_{t=0}^{\infty} (\beta G_C)^t \mathsf{z}_t \left[\frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \log (c_t - \varepsilon c_{t-1}) + \mathsf{j}_t \log h_t - \frac{\tau_t}{1+\eta} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{1+\eta}{1+\xi}} \right]$$

where the term in square brackets represents period utility. With this formulation, the marginal utility of consumption is given by:

$$u_{ct} = \mathsf{z}_t \left(\frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \right) \left(\frac{1}{c_t - \varepsilon c_{t-1}} - \frac{\beta G_C \varepsilon}{c_{t+1} - \varepsilon c_t} \right)$$

the marginal utility of housing is:

$$u_{ht} = \frac{\mathsf{z}_t \mathsf{j}_t}{h_t}$$

and the marginal disutility of working in the goods and housing sector:

$$u_{nct} = \mathsf{z}_t \mathsf{j}_t (1+\eta) n_{ct}^{\xi} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta - \xi}{1+\xi}}$$

$$u_{nht} = \mathsf{z}_t \mathsf{j}_t (1+\eta) n_{ht}^{\xi} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta - \xi}{1+\xi}}$$

Since along the balance growth path (BGP) consumption grows at the rate G_C every quarter, the marginal utility of consumption falls at this rate. Hence the transformed marginal utility $\widetilde{u}_{ct} = u_{ct} G_C^t$ is stationary around the steady state and equal to:

$$\begin{aligned}
\widetilde{u}_{ct} &= G_C^t u_{ct} = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{G_C^t}{c_t - \varepsilon c_{t-1}} - \frac{\beta G_C^{t+1} \varepsilon}{c_{t+1} - \varepsilon c_t} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{1}{\frac{c_t}{G_C^t} - \frac{\varepsilon}{G_C} \frac{c_{t-1}}{G_C^{t-1}}} - \frac{\beta \varepsilon}{\frac{c_{t+1}}{G_C^{t+1}} - \frac{\varepsilon}{G_C} \frac{c_t}{G_C^t}} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{1}{\widetilde{c}_t - \frac{\varepsilon}{G_C} \widetilde{c}_{t-1}} - \frac{\beta \varepsilon}{\widetilde{c}_{t+1} - \frac{\varepsilon}{G_C} \widetilde{c}_t} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{1}{\widetilde{c}_t - \frac{\varepsilon}{G_C} \widetilde{c}_{t-1}} - \frac{\beta \varepsilon}{\widetilde{c}_{t+1} - \frac{\varepsilon}{G_C} \widetilde{c}_t} \right)
\end{aligned}$$

Tranformed consumption, $\widehat{c}_t = c_t / G^t$, and the scaled marginal utility of consumption $\widetilde{u}_{ct}$:

$$\begin{aligned}
\widetilde{u}_c &= \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{1}{1 - \frac{\varepsilon}{G_C}} - \frac{\beta \varepsilon}{1 - \frac{\varepsilon}{G_C}} \right) \frac{1}{\widetilde{c}} = \\
&= \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left(\frac{G_C (1 - \beta \varepsilon)}{G_C - \varepsilon} \right) \frac{1}{\widetilde{c}} = \frac{1}{\widetilde{c}}
\end{aligned}$$

are both constant in steady state.
The marginal utility of housing $u_{ht} = \frac{\mathsf{j}_t \mathsf{z}_t}{h_t}$ declines at the rate G_H. Therefore the transformed marginal utility $\widetilde{u}_{ht} = u_{ht} G_H^t$ is stationary around the steady state and equal to:

$$\widetilde{u}_{ht} = \frac{\mathsf{j}_t \mathsf{z}_t}{\tilde{h}_t}$$

In steady state it is equal to $\overline{\widetilde{u}_h} = \frac{1}{\tilde{h}}$ since both j_t and z_t are equal to one.

Due to the assumptions on preferences and technology hours worked in the two sector are stationary already in the level economy.

The patient household's budget constraint:

$$\begin{aligned} c_t & + \frac{k_{ct}}{\mathsf{A}_{kt}} + k_{ht} + k_{bt} + q_t \left[h_t - (1-\delta_h) h_{t-1}\right] + p_{lt} l_t = \frac{w_{ct}}{X_{wct}} n_{ct} + \frac{w_{ht}}{X_{wht}} n_{ht} \\ + & \quad Div_t - \phi_t + \left(R_{ct} z_{ct} + \frac{1-\delta_k}{\mathsf{A}_{kt}}\right) k_{ct-1} + (R_{ht} z_{ht} + 1 - \delta_k) k_{ht-1} + p_{bt} k_{bt} \\ & + b_t - \frac{R_{t-1} b_{t-1}}{\pi_t} + (R_{lt} + p_{lt}) l_{t-1} - \frac{a(z_{ct})}{\mathsf{A}_{kt}} k_{ct-1} - a(z_{ht}) k_{ht-1} \end{aligned}$$

where the adjustment costs on capital are

$$\phi_t = \frac{\phi_{kc}}{2} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC}\right)^2 \frac{k_{ct-1}}{\Gamma_{Ak}^t} + \frac{\phi_{kh}}{2} \left(\frac{k_{ht}}{k_{ht-1}} - G_C\right)^2 k_{ht-1}$$

where Γ_{Ak} is the gross growth rate of the investment specific technology process in the goods sector and G_{KC} is the BGP gross growth rate of capital in the goods sector. Adjustment costs on capacity utilisation are:

$$\begin{aligned} a(z_{ct}) &= R_c \left(\frac{1}{2} \varpi z_{ct}^2 + (1-\varpi) z_{ct} + \left(\frac{\varpi}{2} - 1\right)\right) \\ a(z_{ht}) &= R_h \left(\frac{1}{2} \varpi z_{ht}^2 + (1-\varpi) z_{ht} + \left(\frac{\varpi}{2} - 1\right)\right) \end{aligned}$$

where R_c and R_h are the steady state levels of the rental rate of capital in, respectively, the goods and the housing sector.
The budget constraint can be transformed as follows:

$$\begin{aligned} & \frac{c_t}{G_C^t} + \frac{k_{ct}}{\mathsf{A}_{kt} G_C^t} + \frac{k_{ht}}{G_C^t} + \frac{k_{bt}}{G_C^t} + \frac{q_t}{G_C^t} \left(\frac{h_t}{G_C^t} - (1-\delta_h) \frac{h_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^{t-1}}\right) + \frac{p_{lt}}{G_C^t} l_t = \frac{w_{ct}}{X_{wct} G_C^t} n_{ct} + \frac{w_{ht}}{X_{wht} G_C^t} n_{ht} \\ & + \frac{Div_t}{G_C^t} - \frac{\phi_t}{G_C^t} + \frac{\Gamma_{AK}^t}{\Gamma_{AK}^{t-1} \Gamma_{AK}} R_{ct} z_{ct} k_{ct-1} \frac{1}{G_C^{t-1} G_C} + (1-\delta_k) \frac{k_{ct-1}}{\mathsf{A}_{kt}} \frac{\mathsf{A}_{kt-1}}{\mathsf{A}_{kt-1}} \frac{1}{G_C^t} \\ & + (R_{ht} z_{ht} + 1 - \delta_k) \frac{k_{ht-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} + p_{bt} \frac{k_{bt}}{G_C^t} + \frac{b_t}{G_C^t} - \frac{R_{t-1}}{\pi_t} \frac{b_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} + \frac{R_{lt} + p_{lt}}{G_C^t} l_{t-1} \\ & - \frac{a(z_{ct})}{\mathsf{A}_{kt}} \frac{k_{ct-1}}{\mathsf{A}_{kt-1}} \frac{\mathsf{A}_{kt-1}}{G_C^t} - a(z_{ht}) \frac{k_{ht-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} \end{aligned}$$

$$
\begin{aligned}
&\frac{c_t}{G_C^t}+\frac{k_{ct}}{\mathsf{A}_{kt}G_C^t}+\frac{k_{ht}}{G_C^t}+\frac{k_{bt}}{G_C^t}+\frac{q_t}{G_C^t}\left(\frac{h_t}{G_C^t}-(1-\delta_h)\frac{h_{t-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^{t-1}}\right)+\frac{p_{lt}}{G_C^t}l_t=\frac{w_{ct}}{X_{wct}G_C^t}n_{ct}+\frac{w_{ht}}{X_{wht}G_C^t}n_{ht}\\
&+\frac{Div_t}{G_C^t}-\frac{\phi_t}{G_C^t}+\frac{R_{ct}\Gamma_{AK}^t}{\Gamma_{AK}G_C}z_{ct}\frac{k_{ct-1}}{\Gamma_{AK}^{t-1}G_C^{t-1}}+(1-\delta_k)\frac{k_{ct-1}}{\mathsf{A}_{kt}}\frac{\mathsf{A}_{kt-1}}{\mathsf{A}_{kt-1}}\frac{1}{G_C^t}\\
&+(R_{ht}z_{ht}+1-\delta_k)\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}+p_{bt}\frac{k_{bt}}{G_C^t}+\frac{b_t}{G_C^t}-\frac{R_{t-1}}{\pi_t}\frac{b_{t-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}+\frac{R_{lt}+p_{lt}}{G_C^t}l_{t-1}\\
&-\frac{a(z_{ct})}{\mathsf{A}_{kt}}\frac{k_{ct-1}}{\mathsf{A}_{kt-1}}\frac{\mathsf{A}_{kt-1}}{G_C^t}-a(z_{ht})\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{c}_t+\frac{\tilde{k}_{ct}}{\mathsf{a}_{kt}}+\tilde{k}_{ht}+\tilde{k}_{bt}+\tilde{q}_t\tilde{h}_t-(1-\delta_h)\tilde{q}_t\frac{\tilde{h}_{t-1}}{G_H}+\tilde{p}_{lt}l_t=\frac{\tilde{w}_{ct}}{X_{wct}}n_{ct}+\frac{\tilde{w}_{ht}}{X_{wht}}n_{ht}+\tilde{Div}_t-\tilde{\phi}_t\\
&+\tilde{R}_{ct}z_{ct}\frac{\tilde{k}_{ct-1}}{G_{KC}}+\frac{(1-\delta_k)}{G_{KC}}\frac{\tilde{k}_{ct-1}}{a_{kt}}+(R_{ht}z_{ht}+1-\delta_k)\frac{\tilde{k}_{ht-1}}{G_C}+p_{bt}\tilde{k}_{bt}\\
&-\frac{a(z_{ct})}{G_{KC}}\frac{\tilde{k}_{ct-1}}{\mathsf{a}_{kt}}-a(z_{ht})\frac{\tilde{k}_{ht-1}}{G_C}+\tilde{b}_t-\frac{R_{t-1}}{\pi_t}\frac{\tilde{b}_{t-1}}{G_C}+\left(\tilde{R}_{lt}+\tilde{p}_{lt}\right)l_{t-1}
\end{aligned}
$$

where the following result, which will be derived later, has been used:

$$G_{KC}^t=\Gamma_{AK}^tG_C^t$$

Adjustment costs for capital can be transformed as follows:

$$\frac{\phi_t}{G_C^t}=\frac{\phi_{kc}}{2}\left(\frac{k_{ct}}{k_{ct-1}}-G_{KC}\right)^2\frac{k_{ct-1}}{G_C^{t-1}G_C\Gamma_{Ak}^{t-1}\Gamma_{Ak}}+\frac{\phi_{kh}}{2}\left(\frac{k_{ht}}{k_{ht-1}}-G_C\right)^2\frac{k_{ht-1}}{G_C^{t-1}G_C}$$

$$\tilde{\phi}_t=\frac{\phi_{kc}}{2G_{KC}}\left(G_{KC}\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}}-G_{KC}\right)^2\tilde{k}_{ct-1}+\frac{\phi_{kh}}{2G_C}\left(G_C\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)^2\tilde{k}_{ht-1}$$

Using the definition of dividends:

$$DIV_t=\left(1-\frac{1}{X_{wct}}\right)w_{ct}n_{ct}+\left(1-\frac{1}{X_{wht}}\right)w_{ht}n_{ht}+\left(1-\frac{1}{X_t}\right)Y_t$$

the terms $\frac{1}{X_{wct}}w_{ct}n_{ct}$ and $\frac{1}{X_{wht}}w_{ht}n_{ht}$ cancel out in the budget constraint so that dividends to the patient households are given by:

$$DIV_t=\left(1-\frac{1}{X_t}\right)Y_t$$

The final expression for the budget constraint is:

$$\tilde{c}_t + \frac{\tilde{k}_{ct}}{\mathsf{a}_{kt}} + \tilde{k}_{ht} + \tilde{k}_{bt} + \tilde{q}_t \tilde{h}_t - (1-\delta_h)\,\tilde{q}_t \frac{\tilde{h}_{t-1}}{G_H} + \tilde{p}_{lt} l_t = \tilde{w}_{ct} n_{ct} + \tilde{w}_{ht} n_{ht} + \left(1 - \frac{1}{X_t}\right)\tilde{Y}_t$$
$$+ \left(\tilde{R}_{ct} z_{ct} + \frac{(1-\delta_k)}{a_{kt}}\right)\frac{\tilde{k}_{ct-1}}{G_{KC}} + (R_{ht} z_{ht} + 1 - \delta_k)\,\frac{\tilde{k}_{ht-1}}{G_C} + p_{bt}\tilde{k}_{bt}$$
$$-\tilde{b}_t + \frac{R_{t-1}}{\pi_t}\frac{\tilde{b}_{t-1}}{G_C} + \left(\tilde{R}_{lt} + \tilde{p}_{lt}\right) l_{t-1} - \frac{a(z_{ct})}{G_{KC}}\frac{\tilde{k}_{ct-1}}{\mathsf{a}_{kt}} - a(z_{ht})\,\frac{\tilde{k}_{ht-1}}{G_C} - \frac{\phi_{kc}}{2G_{KC}}\left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - G_{KC}\right)^2 \tilde{k}_{ct-1}$$
$$+ \frac{\phi_{kh}}{2G_C}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}} - G_C\right)^2 \tilde{k}_{ht-1}$$

The choice variables for the patient household are the following: c_t, h_t, k_{ct}, k_{ht}, b_t, n_{ct}, n_{ht}, k_{bt}, z_{ct} and z_{ht}. The first-order conditions of the patient household's maximisation problem are:

$$\begin{aligned}
u_{ct} q_t &= u_{ht} + \beta G_C E_t\left[u_{ct+1} q_{t+1}(1-\delta_h)\right] \\
u_{ct} &= \beta G_C E_t\left(u_{ct+1} R_t / \pi_{t+1}\right) \\
\frac{u_{ct}}{\mathsf{A}_{kt}}\left(1 + \frac{\partial \phi_{ct}}{\partial k_{ct}}\right) &= \beta G_C E_t\left[u_{ct+1}\left(R_{ct+1} z_{ct+1} - \frac{a(z_{ct})}{\mathsf{A}_{kt}} + \frac{1-\delta_k}{\mathsf{A}_{kt+1}} - \frac{\partial \phi_{ct+1}}{\partial k_{ct}}\right)\right] \\
u_{ct}\left(1 + \frac{\partial \phi_{ht}}{\partial k_{ht}}\right) &= \beta G_C E_t\left[u_{ct+1}\left(R_{ht+1} z_{ht+1} - a(z_{ht}) + 1 - \delta_k - \frac{\partial \phi_{ht+1}}{\partial k_{ht}}\right)\right] \\
u_{nct} &= u_{ct}\frac{w_{ct}}{X_{wct}} \\
u_{nht} &= u_{ct}\frac{w_{ht}}{X_{wht}} \\
u_{ct}(p_{bt} - 1) &= 0 \\
R_{ct} &= \frac{a'(z_{ct})}{\mathsf{A}_{kt}} \\
R_{ht} &= a'(z_{ht}) \\
u_{ct} p_{lt} &= \beta G_C E_t\left[u_{ct+1}(p_{lt+1} + R_{lt+1})\right]
\end{aligned}$$

where we have substitute away the Lagrange multiplier on the budget constraint. These optimality condition must be transformed to take into account the fact the some of the variables are growing over time.
The first order condition with respect to h_t is transformed in the following way:

$$\begin{aligned}
u_{ct} q_t &= u_{ht} + \beta G_C E_t\left(u_{ct+1} q_{t+1}(1-\delta_h)\right) \\
u_{ct} G_C^t \frac{q_t}{G_Q^t} &= \left(u_{ht} G_H^t\right) + \beta G_C E_t\left[\left(u_{ct+1} G_C^t\right)\left(\frac{q_{t+1}}{G_Q^t}\right)(1-\delta_h)\right] \\
u_{ct} G_C^t \frac{q_t}{G_Q^t} &= \left(u_{ht}\frac{G_C^t}{G_Q^t}\right) + \beta G_C E_t\left[\left(u_{ct+1} G_C^{t+1}\frac{1}{G_C}\right)\left(\frac{q_{t+1}}{G_Q^{t+1}} G_Q\right)(1-\delta_h)\right] \\
\widetilde{u}_{ct}\widetilde{q}_t &= \widetilde{u}_{ht} + \beta G_C E_t\left[\widetilde{u}_{ct+1}\widetilde{q}_{t+1}(1-\delta_h)\right]\frac{G_Q}{G_C}
\end{aligned}$$

where G_Q is the BGP growth rate of real house prices whose expression will be derived later.

The transformation that must be applied to the first order condition with respect to lending, b_t, is:

$$
\begin{aligned}
u_{ct} &= \beta G_C E_t \left(\frac{u_{ct+1} R_t}{\pi_{t+1}} \right) \\
u_{ct} G_C^t &= \beta G_C E_t \left(u_{ct+1} \frac{G_C^{t+1}}{G_C} \frac{R_t}{\pi_{t+1}} \right) \\
\widetilde{u}_{ct} &= \beta \widetilde{u}_{ct+1} \frac{R_t}{\pi_{t+1}}
\end{aligned}
$$

The first order condition with respect to k_{ct} is:

$$
\begin{aligned}
u_{ct} \left[\frac{1}{\mathsf{A}_{kt}} + \frac{\phi_{kc}}{G_{KC}} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC} \right) \right] &= \beta G_C E_t \left[u_{ct+1} \left(R_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{A}_{kt+1}} + \frac{1-\delta_k}{\mathsf{A}_{kt+1}} - \right. \right. \\
&+ \left. \frac{\phi_{kc}}{2G_{KC}} \left(\frac{k_{ct+1}^2}{k_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{AK}^{t+1}} \right)
\end{aligned}
$$

It can be transformed in the following way:

$$
\begin{aligned}
G_C^t u_{ct} \left[\frac{1}{\mathsf{A}_{kt}} + \frac{\phi_{kc}}{G_{KC}} \left(G_{KC} \frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - G_{KC} \right) \right] &= \beta G_C E_t \left[G_C^t u_{ct+1} \frac{G_C^{t+1}}{G_C^{t+1}} \left(\frac{\Gamma_{AK}^{t+1}}{\Gamma_{AK}^{t+1}} R_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{A}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{A}_{kt+1}} + \frac{\phi_{kc}}{2G_{KC}} \left(G_{KC}^2 \frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{AK}^{t+1}} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}_{ct} \left[\frac{1}{\mathsf{a}_{kt}} + \phi_{kc} \left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - 1 \right) \right] &= \beta G_C E_t \left[\frac{\tilde{u}_{ct+1} \Gamma_{AK}^t}{G_C \Gamma_{AK}^{t+1}} \left(\tilde{R}_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{a}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{a}_{kt+1}} + \frac{G_{KC}\phi_{kc}}{2} \left(\frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - 1 \right) \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}_{ct} \left[\frac{1}{\mathsf{a}_{kt}} + \phi_{kc} \left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - 1 \right) \right] &= \beta G_C E_t \left[\frac{\tilde{u}_{ct+1}}{G_{KC}} \left(\tilde{R}_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{a}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{a}_{kt+1}} + \frac{G_{KC}\phi_{kc}}{2} \left(\frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - 1 \right) \right) \right]
\end{aligned}
$$

The first order condition with respect to k_{ht} is:

$$
\begin{aligned}
u_{ct} \left[1 + \frac{\phi_{kh}}{G_C} \left(\frac{k_{ht}}{k_{ht-1}} - G_C \right) \right] &= \beta G_C E_t \left[u_{ct+1} \left(R_{ht+1} z_{ht+1} - a\left(z_{ht+1}\right) + 1 - \delta_k - \right. \right. \\
&- \left. \left. \frac{\phi_{kh}}{2G_C} \left(\frac{k_{ht+1}^2}{k_{ht}^2} - G_C^2 \right) \right) \right]
\end{aligned}
$$

This first order condition can be transformed into:

$$\begin{aligned} G_C^t u_{ct}\left[1+\frac{\phi_{kh}}{G_C}\left(G_C\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)\right] &= \beta G_C E_t\left[G_C^t u_{ct+1}\frac{G_C^{t+1}}{G_C^{t+1}}\left(R_{ht+1}z_{ht+1}-a\left(z_{ht+1}\right)+\right.\right.\\ &+ \left.\left.1-\delta_k-\frac{\phi_{kh}}{2G_C}\left(G_C^2\frac{\tilde{k}_{ht+1}^2}{\tilde{k}_{ht}^2}-G_C^2\right)\right)\right] \end{aligned}$$

$$\begin{aligned} \tilde{u}_{ct}\left[1+\phi_{kh}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-1\right)\right] &= \beta G_C E_t\left[\frac{\tilde{u}_{ct+1}}{G_C}\left(R_{ht+1}z_{ht+1}-a\left(z_{ct}\right)+\right.\right.\\ &+ \left.\left.1-\delta_k-\frac{\phi_{kh}G_C}{2}\left(\frac{\tilde{k}_{ht+1}^2}{\tilde{k}_{ht}^2}-1\right)\right)\right] \end{aligned}$$

The first order conditions with respect to u_{nct} and u_{nht} are:

$$\begin{aligned} u_{nct} &= u_{ct}\frac{w_{ct}}{X_{wct}} \\ u_{nht} &= u_{ct}\frac{w_{ht}}{X_{wht}} \end{aligned}$$

which can be transformed as follow:

$$\begin{aligned} u_{nct} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ct}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct}G_C^t\frac{w_{ct}}{X_{wct}G_C^t} \\ u_{nht} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ht}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct}G_C^t\frac{w_{ht}}{X_{wht}G_C^t} \\ u_{nct} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ct}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct}\frac{\tilde{w}_{ct}}{X_{wct}} \\ u_{nht} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ht}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct}\frac{\tilde{w}_{ht}}{X_{wht}} \end{aligned}$$

The first order condition with respect to intermediate inputs k_{bt} is:

$$u_{ct}\left(p_{bt}-1\right)=0$$

which implies that their price is always equal to 1.
The first order conditions with respect to capacity utilisation are:

$$\begin{aligned} R_{ct} &= \frac{a'\left(z_{ct}\right)}{\mathsf{A}_{kt}} \\ R_{ht} &= a'\left(z_{ht}\right) \end{aligned}$$

which are transformed as:

$$\begin{aligned} R_{ct}\mathsf{A}_{kt} &= a'\left(z_{ct}\right) \\ R_{ht} &= a'\left(z_{ht}\right) \end{aligned}$$

$$\begin{aligned}\tilde{R}_{ct} &= \frac{a'(z_{ct})}{\mathsf{a}_{kt}} \\ R_{ht} &= a'(z_{ht})\end{aligned}$$

where we have taken into account the definition $\tilde{R}_{ct} = R_{ct}\Gamma_k^t$.

The first order condition with respect to land l_t

$$u_{ct}p_{lt} = \beta G_C E_t \left(u_{ct+1}\left(p_{lt+1} + R_{lt+1}\right)\right)$$

becomes after transformation:

$$\begin{aligned}u_{ct}G_C^t \frac{p_{lt}}{G_C^t} &= \beta G_C E_t \left[u_{ct+1}G_C^{t+1}\left(\frac{p_{lt+1}}{G_C^{t+1}} + \frac{R_{lt+1}}{G_C^{t+1}}\right)\right] \\ \widetilde{u}_{ct}\widetilde{p}_{lt} &= \beta G_C E_t \left[\widetilde{u}_{ct+1}\left(\widetilde{p}_{lt+1} + \widetilde{R}_{lt+1}\right)\right].\end{aligned}$$

1.2 Impatient households

Lifetime utility is given by:

$$V_t = E_0 \sum_{t=0}^{\infty} \left(\beta' G_C\right)^t \mathsf{z}_t \left[\frac{G_C - \varepsilon'}{G_C - \beta'\varepsilon' G_C} \log\left(c'_t - \varepsilon' c'_{t-1}\right) + \mathsf{j}_t \log h'_t - \frac{\tau_t}{1+\eta'}\left(\left(n'_{ct}\right)^{1+\xi'} + \left(n'_{ht}\right)^{1+\xi'}\right)^{\frac{1+\eta'}{1+\xi'}}\right]$$

With this formulation, the marginal utility of consumption is given by:

$$u'_{ct} = \frac{G_C - \varepsilon'}{G_C - \beta'\varepsilon' G_C}\left(\frac{1}{c'_t - \varepsilon' c'_{t-1}} - \frac{\beta' G_C \varepsilon'}{c'_{t+1} - \varepsilon' c'_t}\right)$$

which can be made stationary using the same transformation employed for the patient households $\widetilde{u}'_{ct} = u'_{ct}G_C^t$. The marginal utility of housing and the marginal disutilities of working are similar to those of the patient households with the exception that the household-specific variables and parameters are denoted with a prime.

The optimality conditions must be transformed to take into account the fact the some of the variables are growing over time. The first order condition with respect to housing, h_t:

$$\begin{aligned}u_{c't}q_t &= u_{h't} + \beta' G_C E_t \left(u_{c't+1}\left(q_{t+1}\left(1-\delta_h\right)\right)\right) + E_t\left(\lambda_t \frac{m q_{t+1}\pi_{t+1}}{R_t}\right) \\ u_{c't}G_C^t \frac{q_t}{G_Q^t} &= u_{h't}\frac{G_C^t}{G_Q^t} + \beta G_C E_t \left(\left(u_{c't+1}G_C^{t+1}\frac{1}{G_C}\right)\left(\frac{q_{t+1}}{G_Q^{t+1}}G_Q\right)\left(1-\delta_h\right)\right) \\ &+ E_t\left(\lambda_t G_C^t \frac{m q_{t+1}\pi_{t+1}}{R_t}\frac{1}{G_Q^t}\frac{G_Q^{t+1}}{G_Q^{t+1}}\right) \\ \widetilde{u}_{c't}\widetilde{q}_t &= \widetilde{u}_{h't} + \beta G_C E_t\left(\widetilde{u}_{c't+1}\widetilde{q}_{t+1}\left(1-\delta_h\right)\right)\frac{G_Q}{G_C} + E_t\left(\tilde{\lambda}_t \frac{m\tilde{q}_{t+1}\pi_{t+1}}{R_t}G_Q\right)\end{aligned}$$

where G_Q is the BGP growth rate of real house prices whose expression will be derived later.

The first order condition with respect to lending, b_t:

$$\begin{aligned} u_{c't} &= \beta G_C E_t \left(\frac{u_{c't+1} R_t}{\pi_{t+1}} \right) + \lambda_t \\ u_{c't} G_C^t &= \beta G_C E_t \left(u_{c't+1} \frac{G_C^{t+1}}{G_C} \frac{R_t}{\pi_{t+1}} \right) + \lambda_t G_C^t \\ \widetilde{u}_{c't} &= \beta \widetilde{u}_{c't+1} \frac{R_t}{\pi_{t+1}} + \tilde{\lambda}_t \end{aligned}$$

The budget constraint:

$$c'_t + q_t \left(h'_t - (1-\delta_h) h'_{t-1} \right) = \frac{w'_{ct}}{X_{wct}} n'_{ct} + \frac{w'_{ht}}{X_{wht}} n'_{ht} + Div'_t + b'_t - \frac{R_{t-1} b'_{t-1}}{\pi_t}$$

$$\frac{c'_t}{G_C^t} + \frac{q_t}{G_C^t} \left(\frac{h'_t}{G_C^t} - (1-\delta_h) \frac{h'_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^{t-1}} \right) = \frac{w'_{ct}}{X_{wct} G_C^t} n'_{ct} + \frac{w'_{ht}}{X_{wht} G_C^t} n'_{ht} + \frac{Div'_t}{G_C^t} + \frac{b'_t}{G_C^t} - \frac{R_{t-1}}{\pi_t} \frac{b'_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t}$$

By substituting the expression for the dividends from the unions the transformed budget constraint becomes:

$$\tilde{c}'_t + \tilde{q}_t \tilde{h}'_t - (1-\delta_h) \tilde{q}_t \frac{\tilde{h}'_{t-1}}{G_H} = \tilde{w}'_{ct} n'_{ct} + \tilde{w}'_{ht} n'_{ht} + \tilde{b}'_t - \frac{R_{t-1}}{\pi_t} \frac{\tilde{b}'_{t-1}}{G_C}$$

The borrowing constraint:

$$b'_t = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{R_t} \right)$$

can be transformed as follows:

$$\frac{b'_t}{G_C^t} = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{G_C^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{G_H^t G_Q^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{G_Q q_{t+1} h'_t \pi_{t+1}}{G_Q^{t+1} G_H^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{G_Q \tilde{q}_{t+1} \tilde{h}'_t \pi_{t+1}}{R_t} \right)$$

1.3 Intermediate goods firms

Wholesale firms solve the following maximization problem:

$$\max \frac{Y_t}{X_t} + q_t IH_t - \left(\sum w_{it} n_{it} + R_{ct} z_{ct} k_{ct-1} + R_{ht} z_{ht} k_{ht-1} + R_{lt} l_{t-1} + p_{bt} k_{bt}\right)$$

The two production technologies are:

$$\begin{aligned} Y_t &= \left[\mathsf{A}_{ct}\left(n_{ct}^{\alpha} n_{ct}^{\prime 1-\alpha}\right)\right]^{1-\mu_c} \left(z_{ct} k_{ct-1}\right)^{\mu_c} \\ IH_t &= \left[\mathsf{A}_{ht}\left(n_{ht}^{\alpha} n_{ht}^{\prime 1-\alpha}\right)\right]^{1-\mu_h-\mu_b-\mu_l} \left(z_{ht} k_{ht-1}\right)^{\mu_h} k_{bt}^{\mu_b} l_{t-1}^{\mu_l} \end{aligned}$$

The first order condition with respect to n_{ct} is:

$$\left(1-\mu_c\right)\alpha \frac{Y_t}{X_t n_{ct}} = w_{ct}$$

which after taking into account that both Y_t and w_{ct} growth at rate G_C along the BGP becomes:

$$\begin{aligned} \left(1-\mu_c\right)\alpha \frac{Y_t}{G_C^t X_t n_{ct}} &= \frac{w_{ct}}{G_C^t} \\ \left(1-\mu_c\right)\alpha \frac{\tilde{Y}_t}{X_t n_{ct}} &= \tilde{w}_{ct} \end{aligned}$$

Similarly for n'_{ct}:

$$\begin{aligned} \left(1-\mu_c\right)\left(1-\alpha\right) \frac{Y_t}{X_t n'_{ct}} &= w'_{ct} \\ \left(1-\mu_c\right)\left(1-\alpha\right) \frac{\tilde{Y}_t}{X_t n'_{ct}} &= \tilde{w}'_{ct} \end{aligned}$$

and for n_{ht}:

$$\begin{aligned} \left(1-\mu_h-\mu_l\right)\alpha \frac{q_t IH_t}{n_{ht}} &= w_{ht} \\ \left(1-\mu_h-\mu_l\right)\alpha \frac{q_t}{G_Q^t} \frac{IH_t}{G_H^t} n_{ht} &= \frac{w_{ht}}{G_C^t} \\ \left(1-\mu_h-\mu_l\right)\alpha \frac{\tilde{q}_t \tilde{IH}_t}{n_{ht}} &= \tilde{w}_{ht} \end{aligned}$$

and for n'_{ht}:

$$\begin{aligned} \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{q_t IH_t}{n'_{ht}} &= w'_{ht} \\ \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{q_t}{G_Q^t} \frac{IH_t}{G_H^t} n'_{ht} &= \frac{w'_{ht}}{G_C^t} \\ \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{\tilde{q}_t \tilde{IH}_t}{n'_{ht}} &= \tilde{w}'_{ht} \end{aligned}$$

The first-order condition with respect to k_{ct-1} is:

$$\mu_c \frac{Y_t}{X_t k_{ct-1}} = R_{ct} z_{ct}$$

which is transformed into:

$$\frac{\mu_c}{X_t}\frac{\frac{Y_t}{G_C^t}}{\frac{k_{ct-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}\frac{1}{\Gamma_{AK}^t}} = R_{ct}\Gamma_{AK}^t z_{ct}$$

$$\frac{\mu_c}{X_t}\frac{\tilde{Y}_t}{\frac{\tilde{k}_{ct-1}}{G_{KC}}} = \tilde{R}_{ct} z_{ct}.$$

Similarly with respect to k_{ht-1} is

$$\begin{aligned}
\mu_h\frac{q_t IH_t}{k_{ht-1}} &= R_{ht} z_{ht} \\
\mu_h\frac{\frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t}}{\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}} &= R_{ht} z_{ht} \\
\mu_h\frac{\tilde{q}_t \tilde{IH}_t}{\frac{\tilde{k}_{ht-1}}{G_C}} &= R_{ht} z_{ht}.
\end{aligned}$$

The first order condition with respect to l_t, after setting $l_t = 1$, is:

$$\begin{aligned}
\mu_l q_t IH_t &= R_{lt} \\
\mu_l\frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t} &= \frac{R_{lt}}{G_C^t} \\
\mu_l \tilde{q}_t \tilde{IH}_t &= \tilde{R}_{lt}
\end{aligned}$$

and with respect to k_{bt}:

$$\begin{aligned}
\mu_b\frac{q_t IH_t}{k_{bt}} &= p_{bt} \\
\mu_b\frac{\frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t}}{\frac{k_{bt}}{G_C^t}} &= p_{bt} \\
\mu_b\frac{\tilde{q}_t \tilde{IH}_t}{\tilde{k}_{bt}} &= p_{bt}
\end{aligned}$$

1.4 Wage stickiness

Patient and impatient households supply their homogeneous labor services to labor unions. There are four unions, two for each sector, each one acting in the interest of either patient or impatient households. The unions differentiate labor services, set nominal wages subject to a Calvo scheme and offer labor services to intermediate labor packers who assemble the differentiated labor services into the homogeneous labor composites n_c, n_h, n'_c and n'_h. The probability of unions being allowed to change nominal wages in each sector is common to both households. Wholesale firms hire labor services from the labor packers. Under partial indexation of nominal wages to past inflation, the wage-setting rules set by the union imply four wage Phillips curves that are isomorphic to the one in the goods sector:[1]

$$\begin{aligned}
\ln \omega_{c,t} - \iota_{wc} \ln \pi_{t-1} &= \beta G_C \left(E_t \ln \omega_{c,t+1} - \iota_{wc} \ln \pi_t\right) - \varepsilon_{wc} \ln \left(X_{wc,t}/X_{wc}\right) \\
\ln \omega'_{c,t} - \iota_{wc} \ln \pi_{t-1} &= \beta' G_C \left(E_t \ln \omega'_{c,t+1} - \iota_{wc} \ln \pi_t\right) - \varepsilon'_{wc} \ln \left(X_{wc,t}/X_{wc}\right) \\
\ln \omega_{h,t} - \iota_{wh} \ln \pi_{t-1} &= \beta G_C \left(E_t \ln \omega_{h,t+1} - \iota_{wh} \ln \pi_t\right) - \varepsilon_{wh} \ln \left(X_{wh,t}/X_{wh}\right) \\
\ln \omega'_{h,t} - \iota_{wh} \ln \pi_{t-1} &= \beta' G_C \left(E_t \ln \omega'_{h,t+1} - \iota_{wh} \ln \pi_t\right) - \varepsilon'_{wh} \ln \left(X_{wh,t}/X_{wh}\right)
\end{aligned}$$

with $\omega_{i,t}$ nominal wage inflation, that is, $\omega_{i,t} = \frac{w_{i,t}\pi_t}{w_{i,t-1}}$ for each sector/household pair, and

$$\begin{aligned}
\varepsilon_{wc} &= \left(1-\theta_{wc}\right)\left(1-\beta G_C \theta_{wc}\right)/\theta_{wc} \\
\varepsilon'_{wc} &= \left(1-\theta_{wc}\right)\left(1-\beta' G_C \theta_{wc}\right)/\theta_{wc} \\
\varepsilon_{wh} &= \left(1-\theta_{wh}\right)\left(1-\beta G_C \theta_{wh}\right)/\theta_{wh} \\
\varepsilon'_{wh} &= \left(1-\theta_{wh}\right)\left(1-\beta' G_C \theta_{wh}\right)/\theta_{wh}
\end{aligned}$$

define the slope of the wage equations.

1.5 Price stickiness

Price stickiness in the consumption-business investment sector is introduced by assuming monopolistic competition at the retail level, implicit costs of adjusting nominal prices following Calvo-style contracts and partial indexation to lagged inflation of those prices that can not be reoptimized. The resulting inflation equation is:

$$\log \pi_t - \iota_\pi \log \pi_{t-1} = \beta \left(E_t \log \pi_{t+1} - \iota_\pi \log \pi_t\right) - \varepsilon_\pi \log \left(\frac{X_t}{X}\right) + \log \mathsf{u}_{p,t}$$

where the parameter ε_π is equal to $\varepsilon_\pi = \frac{(1-\theta_\pi)(1-\beta G_C \theta_\pi)}{\theta_\pi}$.

1.6 Monetary policy

$$R_t = \left(R_{t-1}\right)^{r_R} \left[\pi_t^{r_\pi} \left(\frac{GDP_t}{G_C GDP_{t-1}}\right)^{r_Y}\right]^{1-r_R} \overline{rr}^{1-r_R} \frac{\mathsf{e}_{Rt}}{\mathsf{A}_{St}}$$

where GDP_t is the sum of the value added of the two sectors, that is $GDP_t = Y_t + \overline{q} IH_t + IK_t$

1.7 Market clearing

The market clearing conditions are:

$$\begin{aligned}
C_t + IK_{ct}/\mathsf{A}_{kt} + IK_{ht} + k_{bt} &= Y_t - \frac{\phi_{kc}}{2}\left(\frac{k_{ct}}{k_{ct-1}} - G_{KC}\right)^2 \frac{k_{ct-1}}{\Gamma_{AK}^t} - \frac{\phi_{kh}}{2}\left(\frac{k_{ht}}{k_{ht-1}} - G_C\right)^2 k_{ht-1} \\
h_t + h'_t - \left(1-\delta_h\right)\left(h_{t-1} + h'_{t-1}\right) &= IH_t \\
b_t + b'_t &= 0
\end{aligned}$$

[1] Here we make use of the result that the price-setter stochastic discount factor for nominal payoffs (the ratio between future and current marginal utility of consumption) cancels out in the linearization of the Phillips curve itself, so that the effective discount factor is simply βG_C, rather than $\beta G_C E_t u_{c,t+1}/u_{c,t}$.

which are transformed as follows:

$$\begin{aligned}
\frac{C_t}{G_C^t}+\frac{IK_{ct}/\mathsf{A}_{kt}}{G_C^t}+\frac{IK_{ht}}{G_C^t}+\frac{k_{bt}}{G_C^t} &= \frac{Y_t}{G_C^t}-\frac{\phi_{kc}}{2}\left(\frac{k_{ct}}{k_{ct-1}}\frac{G_{KC}^t}{G_{KC}^t}-G_{KC}\right)^2\frac{k_{ct-1}}{G_C^t\Gamma_{AK}^t}- \\
-\frac{\phi_{kh}}{2}\left(\frac{k_{ht}}{k_{ht-1}}\frac{G_C^t}{G_C^t}-G_C\right)^2\frac{k_{ht-1}}{G_C^t} \\
\frac{h_t}{G_H^t}+\frac{h_t'}{G_HG_H^{t-1}}-(1-\delta_h)\left(\frac{h_{t-1}}{G_HG_H^{t-1}}+\frac{h_{t-1}'}{G_HG_H^{t-1}}\right) &= \frac{IH_t}{G_H^t} \\
\frac{b_t}{G_C^t}+\frac{b_t'}{G_C^t} &= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_t+\frac{I\tilde{K}_{ct}}{\mathsf{a}_{kt}}+I\tilde{K}_{ht}+\tilde{k}_{bt} &= \tilde{Y}_t-\frac{\phi_{kc}}{2G_{KC}}\left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}}-G_{KC}\right)^2\tilde{k}_{ct-1}- \\
-\frac{\phi_{kh}}{2G_C}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)^2\tilde{k}_{ht-1} \\
\tilde{h}_t+\tilde{h}_t'-(1-\delta_h)\left(\frac{\tilde{h}_{t-1}}{G_H}+\frac{\tilde{h}_{t-1}'}{G_H}\right) &= I\tilde{H}_t \\
\tilde{b}_t+\tilde{b}_t' &= 0
\end{aligned}$$

2 Linear deterministic trends

Suppose there are linear deterministic trends in the technologies A_c, A_h and A_k. Let the corresponding gross growth rates be respectively:

$$\gamma_C,\gamma_H,\gamma_K$$

Because of these trends, the variables:

$$Y,c,c',\frac{k_c}{A_k},k_h,k_b,qI$$

all grow at a common rate along the balanced growth path. This result stems from the form of the utility function and the assumption of constant returns to scale in the production functions, which implies common expenditure shares. To compute the net growth rate (x) of Y, we observe from the production function that $x_Y=(1-\mu_c)\gamma_C+\mu_c x_{KC}$. We also know that $x_Y=x_{KC}-\gamma_K$. It then follows that

$$x_Y=\gamma_C+\frac{\mu_c}{1-\mu_c}\gamma_K$$

$$x_{KC}=\gamma_C+\frac{1}{1-\mu_c}\gamma_K$$

$$x_{KH}=\gamma_C+\frac{\mu_c}{1-\mu_c}\gamma_K$$

In order to disentangle q and I separately, we use the formula for I to obtain the steady state growth rate of I as $x_I = (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H + \mu_h x_{KH} + \mu_b x_{KB}$.
Hence the steady state growth rate of I is:

$$x_I = (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H + \mu_h \left(\gamma_C + \frac{\mu_c}{1-\mu_c}\gamma_K\right) + \mu_b \left(\gamma_C + \frac{\mu_c}{1-\mu_c}\gamma_K\right)$$

$$x_I = x_H = (\mu_h + \mu_b)\,\gamma_C + \frac{(\mu_h + \mu_b)\,\mu_c}{1-\mu_c}\gamma_K + (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H$$

and the growth rate of q is

$$x_Q = (1 - \mu_h - \mu_b)\,\gamma_C + \frac{(1 - \mu_h - \mu_b)\,\mu_c}{1-\mu_c}\gamma_K - (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H$$

$$x_Q = x_Y - x_I$$

3 Steady state of the model

We are interested in finding the steady state of the transformed model. In the transformed model, each variable is scaled by its long-run growth rate, e.g.

$$\widetilde{c}_t = \frac{c_t}{G_C^t}$$
$$\widetilde{c}_{t-1} = \frac{c_{t-1}}{G_C^{t-1}}$$

hence in each equation we perform the necessary replacements such as the following:

$$c_t = \widetilde{c}_t G_C^t$$
$$c_{t-1} = \widetilde{c}_{t-1} G_C^{t-1}$$
$$q_t = \widetilde{q}_t G_Q^t$$
$$u_{ct} = \widetilde{u}_{ct} G^{-t}$$

3.1 Calculations

Marginal utility of consumption and housing are equal, respectively, to $1/c$ and j/h in steady state. From the transformed consumption Euler equation:

$$u_{ct} = \beta G_C u_{ct+1} \frac{R_t}{\pi_{t+1}}$$

the G_C term disappears and we can derive the steady state level of the real interest rate once we have imposed $\overline{\pi} = 1$:

$$R = \frac{1}{\beta}$$

From the Euler equations for the two capital stocks we can derive the steady state values for the rental rates:

$$R_{kc} = \frac{\Gamma_K}{\beta} - (1 - \delta_k)$$
$$R_{kh} = \frac{1}{\beta} - (1 - \delta_k)$$
$$r \equiv \frac{R}{G_C} - 1$$

Combining the Euler equation for k_c and the expression for R_{kc} (from the optimal demand for capital by firms in the good sector) the following ratio is obtained:

$$\zeta_0 = \frac{k_c}{Y} = \left(\frac{\beta G_{KC} \mu_c}{\Gamma_K - \beta (1 - \delta_{kc})} \right) \frac{1}{X}$$

Combining the Euler equation for k_h and the expression for R_{kh} from the optimal demand for capital by firms in the good sector the following ratio is obtained:

$$\zeta_1 = \frac{k_h}{qI} = \frac{\beta G_C \mu_h}{1 - \beta (1 - \delta_{kh})}$$

From the Euler equation for h:

$$\zeta_2 = \frac{qh}{c} = \frac{j}{1-\beta G_Q(1-\delta_h)}$$

while from the Euler equation for h' and b':

$$\begin{aligned}
\zeta_3 &= \frac{j}{1-\beta' G_Q(1-\delta_h) - G_Q(\beta-\beta')m} \\
\lambda &= \frac{1-\beta'/\beta}{c'} \\
f &= \frac{X-1}{X}Y
\end{aligned}$$

For land, let $l=1$, so that

$$R_l = \mu_l qI$$

The following equations describe the steady state (using $b+b'=0$, where $b = mG_Q qh'/R$ and steady state repayment is $\left(\frac{R}{G_C}-1\right)b$, so that repayment equals $\left(\frac{R}{G_C}-1\right)\frac{mG_Q}{R}qh' = \zeta_4 qh'$):
Define the adjusted depreciation rates:

$$\begin{aligned}
\delta_h' &= 1-\frac{1-\delta_h}{G_H} \\
\delta_k' &= 1-\frac{1-\delta_k}{G_{KC}}
\end{aligned}$$

From the above ratios and using the budget constraints of the two types of households, we have:

$$\begin{aligned}
k_c &= \zeta_0 Y \\
k_h &= \zeta_1 qI \\
qh &= \zeta_2 c \\
qh' &= \zeta_3 c' \\
\delta_h'(qh+qh') &= qI \\
c+c'+\delta_k'(k_c+k_h) &= Y \\
c+\delta_h' qh &= f + rk_c + rk_h + \mu_l qI + \sum wn + \zeta_4 qh' + \text{div} \\
c' + \delta_h' qh' &= \sum wn - \zeta_4 qh' + \text{div}
\end{aligned}$$

Simple algebra yields:

$$\begin{aligned}
\delta_h'(\zeta_2 c + \zeta_3 c') &= qI \\
c+c'+\delta_k'(\zeta_0 Y + \zeta_1 qI) &= Y \\
c+\delta_h'\zeta_2 c &= f + r\zeta_0 Y + r\zeta_1 qI + \mu_l qI + \sum wn + \zeta_4\zeta_3 c' + div \\
c' + \delta_h'\zeta_3 c' &= \sum wn - \zeta_4\zeta_3 c' + \text{div}
\end{aligned}$$

The equations in labor market satisfy from the demand side:

$$\begin{aligned}
(1-\mu_c)\,\alpha\frac{Y}{Xn_c} &= w_c \\
(1-\mu_c)\,(1-\alpha)\,\frac{Y}{Xn_c'} &= w_c' \\
(1-\mu_h-\mu_b-\mu_l)\,\alpha\frac{qI}{n_h} &= w_h \\
(1-\mu_h-\mu_b-\mu_l)\,(1-\alpha)\,\frac{qI}{n_h'} &= w_h'
\end{aligned}$$

To compute the steady state, we simply need to know the total wage bill plus union dividends earned by each group, which equals

$$\begin{aligned}
w_cn_c+w_hn_h &= \alpha\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right) \\
w_c'n_c'+w_h'n_h' &= (1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)
\end{aligned}$$

Using $\phi=(X-1)/X$, we have

$$\delta_h'\left(\zeta_2c+\zeta_3c'\right)=qI$$
$$c+c'+\delta_k'\left(\zeta_0Y+\zeta_1qI\right)=Y$$
$$c+\delta_h'\zeta_2c=\phi Y+r\zeta_0Y+r\zeta_1qI+\mu_lqI+\alpha\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)+\zeta_4\zeta_3c'$$
$$c'+\delta_h'\zeta_3c'=(1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)-\zeta_4\zeta_3c'$$

Eliminating one redundant equation (for example the second) and using the formula for qI

$$\begin{aligned}
c+\delta_h\zeta_2c &= (\phi+r\zeta_0)\,Y+r\zeta_1\delta_h\left(\zeta_2c+\zeta_3c'\right)+\alpha\left(\frac{(1-\mu_c)\,Y}{X}+(1-\mu_b-\mu_h-\mu_l)\,\delta_h\left(\zeta_2c+\zeta_3c'\right)\right)+\zeta_4\zeta_3c' \\
c'+\delta_h\zeta_3c' &= (1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,\delta_h'\left(\zeta_2c+\zeta_3c'\right)\right)-\zeta_4\zeta_3c'
\end{aligned}$$

Hence the consumption-output ratios c/Y and c'/Y solve:

$$\begin{aligned}
&\left(1+\delta_h'\zeta_2\left(1-r\zeta_1-\mu_l-\alpha\left(1-\mu_b-\mu_h-\mu_l\right)\right)\right)c-\left(\left(r\zeta_1+\mu_l+\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\delta_h'\zeta_3+\zeta_4\zeta_3\right)c' \\
= &\left(\frac{X-1}{X}+r\zeta_0X+\alpha\frac{(1-\mu_c)}{X}\right)Y \\
&\left(1+\delta_h'\zeta_3-(1-\alpha)\,(1-\mu_h-\mu_b-\mu_l)\,\delta_h'\zeta_3+\zeta_4\zeta_3\right)c'-(1-\alpha)\,(1-\mu_h-\mu_b-\mu_l)\,\delta_h'\zeta_2c \\
= &(1-\alpha)\,(1-\mu_c)\,\frac{1}{X}Y
\end{aligned}$$

Solve for c/Y, c'/Y and qI/Y. Defining the following variables:

$$\begin{aligned}
\chi_1 &= 1+\delta_h'\zeta_2\left(1-r\zeta_1-\mu_l-\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\\
\chi_2 &= \left(r\zeta_1+\mu_l+\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\delta_h'\zeta_3+\zeta_4\zeta_3\\
\chi_3 &= \frac{X-1}{X}+r\zeta_0 X+\alpha\frac{(1-\mu_c)}{X}\\
\chi_4 &= 1+\delta_h'\zeta_3-(1-\alpha)\left(1-\mu_h-\mu_b-\mu_l\right)\delta_h'\zeta_3+\zeta_4\zeta_3\\
\chi_5 &= (1-\alpha)\left(1-\mu_h-\mu_b-\mu_l\right)\delta_h'\zeta_2\\
\chi_6 &= (1-\alpha)\left(1-\mu_c\right)\frac{1}{X}
\end{aligned}$$

delivers the following solution:

$$\begin{aligned}
\frac{c}{Y} &= \frac{\chi_3\chi_4+\chi_2\chi_6}{\chi_1\chi_4-\chi_2\chi_5}\\
\frac{c'}{Y} &= \frac{\chi_1\chi_6+\chi_3\chi_5}{\chi_1\chi_4-\chi_2\chi_5}\\
\frac{qI}{Y} &= \delta_h'\left(\zeta_2 c+\zeta_3 c'\right)
\end{aligned}$$

3.2 Levels

In order to compute the levels of the variables in steady state we need to find first the value of hours worked. It is useful to normalize τ to 1. The labor market equilibrium is of the kind:

$$\begin{aligned}
(1-\mu_c)\,\alpha\frac{Y}{X} &= X_w c\left(n_c^{1+\xi}+n_h^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} n_c^{1+\xi}\\
\left(1-\mu_h-\mu_b-\mu_l\right)\alpha qI &= X_w c\left(n_c^{1+\xi}+n_h^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} n_h^{1+\xi}
\end{aligned}$$

so that the ratio of hours worked is:

$$\frac{n_h}{n_c}=\left(\frac{\left(1-\mu_h-\mu_b-\mu_l\right)qIX}{(1-\mu_c)\,Y}\right)^{\frac{1}{1+\xi}}$$

plug back to get

$$(1-\mu_c)\,\alpha\frac{Y}{Xc}=X_w\left(1+\frac{\left(1-\mu_h-\mu_b-\mu_l\right)qIX}{(1-\mu_c)\,Y}\right)^{\frac{\eta-\xi}{1+\xi}} n_c^{1+\eta}$$

knowing $\frac{Y}{c}$ and $\frac{qI}{Y}$, this can be solved for n_c, and consequently for all the variables of the model:

$$n_c=\left(\frac{(1-\mu_c)\,\alpha\frac{Y}{X_w X c}}{\left(1+\frac{(1-\mu_h-\mu_b-\mu_l)qIX}{(1-\mu_c)Y}\right)^{\frac{\eta-\xi}{1+\xi}}}\right)^{\frac{1}{1+\eta}}$$

Similar formulas apply to n_h, n_c' and n_h'. Once we know the levels of hours worked by the two households in the two sectors, we can compute Y, c, c', k_c, k_h and the product qI. To find q and I separately we use:

$$k_b=\mu_b qI$$

and

$$I = \left(A_h n_h^{\alpha} n_h'^{1-\alpha}\right) (\zeta_1 q)^{\frac{\mu_h}{1-\mu_h-\mu_b-\mu_l}} (\mu_b q)^{\frac{\mu_h}{1-\mu_h-\mu_b-\mu_l}}$$

Let qI be equal to θ, then:

$$\begin{aligned} qI &= \theta \\ I &= \left(A_h n_h^{\alpha} n_h'^{1-\alpha}\right)^{1-\mu_h-\mu_l} (\zeta_1 \theta)^{\mu_h} (\mu_b \theta)^{\mu_b} \end{aligned}$$

Given the values of hours worked, we can use the production function in the goods sector to compute output Y:

$$Y =$$

The levels of capital stock in the two sectors are respectively:

$$\begin{aligned} k_c &= \zeta_0 Y \\ k_c &= \zeta_1 Q I \end{aligned}$$

the levels of consumption of the two agents:

$$\begin{aligned} c &= \left(\frac{\chi_3\chi_4 + \chi_2\chi_6}{\chi_1\chi_4 - \chi_2\chi_5}\right) Y \\ c' &= \left(\frac{\chi_1\chi_6 + \chi_3\chi_5}{\chi_1\chi_4 - \chi_2\chi_5}\right) Y \end{aligned}$$

and their stock of housing:

$$\begin{aligned} h &= \zeta_2 \frac{c}{q} \\ h' &= \zeta_3 \frac{c'}{q} \end{aligned}$$

Finally, the level of loans is:

$$b = m\, q\, G_Q\, \frac{h'}{r}$$