Essential Algebra of ISOM paper

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1. The Core Model

1.1. Household Savers

Savers solve

$$\max \sum_{t=0}^{\infty} \beta_H^t \frac{(C_{H,t})^{1-\sigma_H}}{1-\sigma_H} - \eta_1 \frac{N_{H,t}^{1+\eta_2}}{1+\eta_2}$$

subject to:

$$C_{H,t} + D_t + B_{H,t} + K_{H,t} = R_{D,t} D_{t-1} + R_{B,t} B_{H,t-1} + (R_{KH} + 1 - \delta) K_{H,t-1} + W_t N_t \tag{1}$$

yielding

$$C_{H,t}^{-\sigma_H} = \beta_H R_{D,t+1} C_{H,t+1}^{-\sigma_H} \tag{2}$$

$$C_{H,t}^{-\sigma_H} = \beta_H R_{B,t+1} C_{H,t+1}^{-\sigma_H} \tag{3}$$

$$W_t C_{H,t}^{-\sigma_H} = \eta_1 N_{H,t}^{\eta_2} \tag{4}$$

$$C_{H,t}^{-\sigma_H} = \beta_H (R_{KH,t+1} + 1 - \delta) C_{H,t+1}^{-\sigma_H} \tag{5}$$

Combine hh focs, set $\phi = 0$, then

$$E_t R_{B,t+1} = E_t R_{D,t+1}$$

up to first order, household is willing to hold both as a residual claimant so long as $ER_B = ER_D$

1.2. Bankers

They solve

$$\max \sum_{t=0}^{\infty} \beta_B^t \frac{(C_{B,t})^{1-\sigma_B}}{1-\sigma_B}$$

subject to:

$$C_{B,t} + R_{D,t}D_{t-1} + L_t + B_{B,t} = D_t + R_{L,t}L_{t-1} + R_{B,t}B_{B,t-1} \tag{6}$$
$$D_t - \rho_D D_{t-1} \leq (1-\rho_D)(\gamma_L L_t + \gamma_B B_{B,t}) + \rho_D (L_t + B_{B,t} - (L_{t-1} + B_{B,t-1})) \tag{7}$$

choose C, D, L, B_B to get:

$$\mu_{B,t} = C_{B,t}^{-\sigma_B}$$
$$(1-\lambda_{B,t})\mu_{B,t} = \beta_B (R_{D,t+1} - \rho_D \lambda_{B,t+1})\mu_{B,t+1} \tag{8}$$
$$(1-(\gamma_L(1-\rho_D)+\rho_D)\lambda_{B,t})\mu_{B,t} = \beta_B (R_{L,t+1} - \rho_D \lambda_{B,t+1})\mu_{B,t+1} \tag{9}$$
$$(1-(\gamma_B(1-\rho_D)+\rho_D)\lambda_{B,t})\mu_{B,t} = \beta_B (R_{B,t+1} - \rho_D \lambda_{B,t+1})\mu_{B,t+1} \tag{10}$$

Combine to get

$$\begin{aligned} D_t &\leq \gamma_L L_t + \gamma_B B_{B,t} \\ R_{L,t+1} &= R_{B,t+1}\frac{1-\gamma_L\lambda_{B,t}}{1-\gamma_B\lambda_{B,t}} \end{aligned}$$

For $\gamma_L < \gamma_B = 1$, a drop in λ_B reduces $\frac{1-\gamma_L\lambda_{B,t}}{1-\gamma_B\lambda_{B,t}}$, hence if ER_L rises ER_B might fall, thus triggering a reduction in B_B.

1.3. Entrepreneurs

Entrepreneurs transform loan into capital using one-for-one technology and can convert loan back into consumption

$$\max \sum_{t=0}^{\infty} \beta_E^t \frac{(C_{E,t})^{1-\sigma_E}}{1-\sigma_E}$$

subject to:

$$C_{E,t} + K_t + R_{L,t}L_{t-1} = R_{K,t}K_{t-1} + (1-\delta)K_{t-1} + L_t \tag{11}$$
$$L_t = \rho_E L_{t-1} + m(1-\rho_E)K_t \qquad (\lambda_{E,t}) \tag{12}$$

yielding:

$$(1-\lambda_{E,t})C_{E,t}^{-\sigma_E} = \beta_E (R_{L,t+1} - \rho_E \lambda_{E,t+1})C_{E,t+1}^{-\sigma_E} \tag{13}$$
$$(1-\lambda_{E,t}(1-\rho_E)m)C_{E,t}^{-\sigma_E} = \beta_E (R_{K,t+1} + 1 - \delta)C_{E,t+1}^{-\sigma_E} \tag{14}$$

1.4. Firms

$$Y_t = A_t K_{Ht-1}^{\alpha(1-\mu)} K_{Et-1}^{\alpha\mu} N_t^{1-\alpha} \tag{15}$$
$$\alpha\mu Y_t = R_{KE,t}K_{Et-1} \tag{16}$$
$$\alpha(1-\mu)Y_t = R_{KH,t}K_{Ht-1} \tag{17}$$
$$(1-\alpha)Y_t = W_t N_t \tag{18}$$

1.5. Government

The budget constraint is simply

$$G_t + R_{B,t}B_{t-1} = B_t + T_t \tag{19}$$
$$T_t = \overline{T}Y_t + g(B_t, B_{t-1}) \tag{20}$$
$$G_t = \overline{G}Y_t \tag{21}$$
$$B_t = B_{B,t} + B_{H,t} \tag{22}$$

Count $Y, C_B, C_E, C_H, K, L, D, B, B_B, B_H, T, G, \lambda_B, \lambda_E, R_B, R_D, R_K, R_L, W, \mu$

2. Steady State Algebra

Prices

$$R_B = \frac{1}{\beta_H} \tag{1}$$

$$R_D = \frac{1}{\beta_H} \tag{2}$$

$$\lambda_B = \frac{1-\beta_B R_B}{1-\beta_B \rho_D} \tag{3}$$

$$R_L = \frac{1}{\beta_B} - \frac{(1-\beta_B)\rho_D + (1-\rho_D)\gamma_L}{\beta_B} \frac{1-\beta_B R_H}{1-\beta_B \rho_D} \tag{4}$$

$$\lambda_E = \frac{1-\beta_E R_L}{1-\beta_E \rho_E} \tag{5}$$

$$R_{KE} = \frac{1}{\beta_E}\left(1-\lambda_E\left(1-\rho_E\right)m\right) - (1-\delta) \tag{6}$$

$$R_{KH} = \frac{1}{\beta_H} - (1-\delta) \tag{7}$$

Quantities

$$\begin{aligned}
D &= \gamma_L L + \gamma_B B_B \\
C_H &= (1-\alpha)Y + (R_D-1)D + (R_B-1)B_H + \alpha(1-\mu)Y - \delta K_H - T \\
C_B &= (R_L-1)L + (R_B-1)B_B - (R_D-1)D \\
C_E &= \alpha\mu Y - \delta K_E - (R_L-1)L \\
G-T &= -(R_B-1)B
\end{aligned}$$

Sum across

$$C + G - T = Y - T - \delta K$$

From the production side

$$K_E = \frac{\alpha\mu}{R_{KE}} Y = \kappa_E Y$$

$$K_H = \frac{\alpha(1-\mu)}{R_{KH}} Y = \kappa_H Y$$

$$L = m K_E$$

$$G - T = -(R_B-1)(B_B+B_H)$$

$$B_H = s_H \frac{\overline{T}-\overline{G}}{1/\beta_H - 1} Y$$

$$B_B = (1-s_H)\frac{\overline{T}-\overline{G}}{1/\beta_H - 1} Y$$

$$\frac{D}{Y} = \gamma_L \frac{L}{Y} + \gamma_B \frac{B_B}{Y}$$

$$\frac{C_H}{Y} = (1-\alpha) + \alpha(1-\mu)\frac{K_H}{Y} - \delta\frac{K_H}{Y} + (R_D-1)\frac{D}{Y} + (R_B-1)\frac{B_H}{Y} - \frac{T}{Y} = c_H Y$$

$$\frac{C_E}{Y} = \alpha\mu - \delta\kappa_E - (R_L-1)\frac{L}{Y}$$

$$\frac{C_B}{Y} = (R_L-1)\frac{L}{Y} + (R_B-1)\frac{B_B}{Y} - (R_D-1)\frac{D}{Y}$$

Determine N_H

$$
\begin{gathered}
(1-\alpha)\frac{Y}{N_H}C_{H,t}^{-\sigma_H} = \eta_1 N_{H,t}^{\eta_2} \\
(1-\alpha)\frac{Y}{(c_H Y)^{\sigma_H}} = \eta_1 N_H^{1+\eta_2} \\
N_H = \left(\frac{(1-\alpha)\,Y^{1-\sigma_H}}{\eta_1 c_H^{\sigma_H}}\right)^{\frac{1}{1+\eta_2}} \\
Y = (\kappa_E Y)^{\alpha\mu}(\kappa_H Y)^{\alpha(1-\mu)}\left(\frac{(1-\alpha)\,Y^{1-\sigma_H}}{\eta_1 c^{\sigma_H}}\right)^{\frac{1-\alpha}{1+\eta_2}} \\
Y = (\kappa_E)^{\frac{\alpha\mu}{1-\alpha-\frac{(1-\sigma_H)(1-\alpha)}{1+\eta_2}}}(\kappa_H)^{\frac{\alpha(1-\mu)}{1-\alpha-\frac{(1-\sigma_H)(1-\alpha)}{1+\eta_2}}}\left(\frac{1-\alpha}{\eta_1 c^{\sigma_H}}\right)^{\frac{1-\alpha}{(1-\alpha)(1+\eta_2)-(1-\sigma_H)}} \\
\text{plug back } Y \text{ to get } N_H
\end{gathered}
$$

With Y in hand, all levels follow, see formulas in def_parm_m1

3. Demand and supply curves

3.1. Market for D

Supply

$$C_{H,t}^{-\sigma_H}\left(1+\phi_{DH}\left(\frac{D_t-D_{t-1}}{D_{t-1}}\right)\right)=\beta_H R_{D,t+1}C_{H,t+1}^{-\sigma_H}$$

$$d_t=d_{t-1}+\frac{1}{\phi_{DH}}\left(r_{D,t+1}+\sigma_H\left(c_{H,t}-c_{H,t+1}\right)\right) \tag{d\_s}$$

Demand (assume $\rho_D=0$)

$$\left(1-\lambda_{B,t}-\phi_{DB}\left(\frac{D_t-D_{t-1}}{D_{t-1}}\right)\right)C_{B,t}^{-\sigma_B}=\beta_B R_{D,t+1}C_{B,t+1}^{-\sigma_B}$$

$$\log\left(1-\lambda-\phi_{DB}\left(\frac{D_t-D_{t-1}}{D_{t-1}}\right)\right)-\sigma_B C_{B,t}=\log\beta+\log R_{D,t+1}-\sigma_B C_{B,t+1}$$

$$d_t=d_{t-1}-\frac{1}{\phi_{DB}}\left(r_{D,t+1}+\sigma_B\left(c_{B,t}-c_{B,t+1}\right)+\frac{\widetilde{\lambda}_{B,t}}{1-\lambda_B}\right) \tag{d\_d}$$

where $\widetilde{\lambda}_{B,t}=\lambda_t-\lambda_{SS}$

Consider now a technology shock. Deposit supply rises if current consumption rises more than expected consumption.

Deposit demand FALLS if current consumption rises more than expected consumption, and if the deposit rate rises. A tightening of the bank's constraint (rise in λ) also constrains bank's ability to demand funds, hence it leads to a reduction in the demand for deposits.

TFP shock: rise in expected consumption shifts deposit demand inwards.

3.2. Market for B_H

Supply (from households)

$$b_{H,t}=b_{H,t-1}+\frac{1}{\phi_{BH}}\left(r_{B,t+1}+\sigma_H\left(c_{H,t}-c_{H,t+1}\right)\right) \tag{bh\_s}$$

Supply (from banks)

$$\left(1-\gamma_B\lambda_{B,t}\right)\mu_{B,t}=\beta_B\left(R_{B,t+1}\right)\mu_{B,t+1}$$

$$\rho=0,\gamma=1$$

$$\log\left(1-\gamma_B\lambda+\phi_{BB}\left(\frac{B_t-B_{t-1}}{B_{t-1}}\right)\right)+\log\mu=\log\beta+\log R'+\log\mu'$$

$$b_{B,t}=b_{B,t-1}+\frac{1}{\phi_{BB}}\left(r_{B,t+1}+\sigma_B\left(c_{B,t}-c_{B,t+1}\right)+\gamma_B\frac{\widetilde{\lambda}_{B,t}}{1-\gamma_B\lambda_B}\right) \tag{bb\_s}$$

Total supply is a weighted average of $b_{H,t}$ and $b_{B,t}$

Banks will want to hold more bonds when λ rises, since it is an asset and provides liquidity.

Demand from government

$$\begin{aligned}
B_t &= G_t+R_{B,t}B_{t-1}-T_t\\
T_t &= \tau Y_t+\theta\left(B_t-B_{t-1}\right)\\
B_t &= gY_t+R_{B,t}B_{t-1}-\tau Y_t-\theta\left(B_t-B_{t-1}\right)\\
\left(1+\theta\right)B_t &= gY_t+R_{B,t}B_{t-1}-\tau Y_t+\theta B_{t-1}\\
\left(1+\theta\right)dB_t &= gdY_t+RdB_{t-1}+BdR_{B,t}-\tau dY_t+\theta dB_{t-1}\\
\left(1+\theta\right)\frac{dB_t}{B}\frac{B}{Y} &= gy_t+R\frac{dB_{t-1}}{B}\frac{B}{Y}+\frac{RB}{Y}\frac{dR_{B,t}}{R}-\tau y_t+\theta\frac{B}{Y}\frac{dB_{t-1}}{B}\\
\left(1+\theta\right)\frac{B}{Y}b_t &= \left(g-\tau\right)y_t+\left(R+\theta\right)\frac{B}{Y}b_{t-1}+\frac{RB}{Y}r_{B,t}
\end{aligned}$$

Hence rise in r implies a rise in demand for b with an elasticity that is given by

$$T-G=(R-1)B-->\tau-g=(R-1)\frac{B}{Y}$$

$$\begin{aligned}
(1+\theta)\frac{B}{Y}b_t &= (g-\tau)y_t+(R+\theta)\frac{B}{Y}b_{t-1}+\frac{RB}{Y}r_{B.t} \\
(1+\theta)\frac{\tau-g}{R-1}b_t &= (g-\tau)y_t+(R+\theta)\frac{\tau-g}{R-1}b_{t-1}+R\frac{\tau-g}{R-1}r_{B.t} \\
(1+\theta)b_t &= -(R-1)y_t+(R+\theta)b_{t-1}+Rr_{B.t} \\
b_t &= -\frac{R-1}{1+\theta}y_t+\frac{R+\theta}{1+\theta}b_{t-1}+\frac{R}{1+\theta}r_{B.t}-\frac{d\varepsilon_t}{B}
\end{aligned}$$

Note that this equation can be forwarded one period and written as

$$b_t=\frac{1+\theta}{R+\theta}b_{t+1}+\frac{R-1}{R+\theta}y_{t+1}-\frac{R}{R+\theta}r_{B,t+1}+\frac{1+\theta}{R+\theta}\frac{d\varepsilon_{t+1}}{B_{SS}} \qquad \text{(b\_d)}$$

3.3. Market for L

Supply (from banks)

$$l_t=l_{t-1}+\frac{1}{\phi_{LB}}\left(r_{L,t+1}+\sigma_B\left(c_{B,t}-c_{B,t+1}\right)+\gamma_L\frac{\widetilde{\lambda}_{B,t}}{1-\gamma_L\lambda_B}\right) \qquad \text{(l\_s)}$$

Demand (from E)

$$\begin{aligned}
\left(1-\phi\frac{L_t-L_{t-1}}{L_{t-1}}-\lambda_{E,t}\right)C_{E,t}^{-\sigma_E} &= \beta_E\left(R_{L,t+1}-\rho_E\lambda_{E,t+1}\right)C_{E,t+1}^{-\sigma_E} \\
l_t=l_{t-1}-\frac{1}{\phi_L}\left(r_{L,t+1}+\sigma_E\left(c_{E,t}-c_{E,t+1}\right)+\frac{\widetilde{\lambda}_{E,t}}{1-\lambda_E}\right) & \qquad \text{(l_d)}
\end{aligned}$$

3.4. Market for K_E

Supply (from E)

$$\left(1+\phi\frac{K_t-K_{t-1}}{K_{t-1}}-\lambda_{E,t}\left(1-\rho_E\right)m\right)C_{E,t}^{-\sigma_E}=\beta_E\left(R_{K,t+1}+1-\delta\right)C_{E,t+1}^{-\sigma_E}$$

$$\log\left(1+\phi\frac{K_t-K_{t-1}}{K_{t-1}}-\lambda_{E,t}\left(1-\rho_E\right)m\right)-\sigma_E\log C_{E,t}=\log\beta_E+\log\left(R_{K,t+1}+1-\delta\right)-\sigma_E\log C_{E,t+1}$$

$$-m\frac{\lambda_{E,t}-\lambda_E}{1-\lambda_E m}+\phi\left(k_t-k_{t-1}\right)-\sigma_E c_t=\frac{R_{K,t+1}-R_K}{R_K+1-\delta}-\sigma_E c_{t+1}$$

$$k_{E,t}=k_{E,t-1}+\frac{1}{\phi_K}\left(\frac{R_{K,t+1}-R_K}{R_K+1-\delta}+\sigma_E\left(c_{E,t}-c_{E,t+1}\right)+\frac{\widetilde{\lambda}_{E,t}}{1-\lambda_E}\right) \qquad \text{(k\_s)}$$

Demand (from firm)

$$\begin{aligned}
\alpha\mu A_t K_{E,t-1}^{\alpha\mu-1}K_{H,t-1}^{\alpha(1-\mu)} &= R_{KE,t} \\
r_{KE,t} &= a_t-\left(1-\alpha\mu\right)k_{E,t-1}+\alpha\left(1-\mu\right)k_{H,t-1} \\
k_{E,t} &= \frac{1}{1-\alpha\mu}\left(a_{t+1}+\alpha\left(1-\mu\right)k_{H,t}-r_{KE,t+1}\right) \qquad \text{(k_d)}
\end{aligned}$$

4. Loose Ends

1. Varying Maturity of Debt
   Follow http://public.econ.duke.edu/~fb36/Papers_Francesco_Bianchi/BianchiIlut_Policy_Mix.pdf
   Instead of
   $$C_{H,t} + D_t + B_{H,t} = R_{D,t}D_{t-1} + R_{B,t}B_{H,t-1} + W_t N_t$$
   write budget as
   $$C_{H,t} + P_t D_t + B_{H,t} = (1 + \rho P_t)\, D_{t-1} + R_{B,t}B_{H,t-1} + W_t N_t$$
   where ρ controls the maturity. The Euler is
   $$\begin{aligned} \frac{1}{C}P_t &= \beta\left(1+\rho P_{t+1}\right)\frac{1}{C'} \\ \frac{1}{C} &= \beta\frac{1+\rho P_{t+1}}{P_t}\frac{1}{C'} \end{aligned}$$
   If
   $$\begin{aligned} \frac{1}{P} &= R \\ \frac{P'}{P} &= \frac{R}{R'} \\ \frac{1}{C} &= \beta\left(R+\rho\frac{R}{R'}\right)\frac{1}{C'} \\ \rho &= 0->\frac{1}{C}=\beta\left(R\right)\frac{1}{C'} \\ \rho &= 1->\frac{1}{C}=\beta\left(R+\frac{R}{R'}\right)\frac{1}{C'} \end{aligned}$$

5. Dynamic Responses to a Default Shock

In this experiment, the government does not pay back its depositors.

1. When banks hold little government debt, the loss is borne by the households, but is almost a wash at the aggregate level, for standard ricardian reasons.

2. When banks hold a lot of government debt, the loss is borne by the bank, which ends up absorbing the higher debt financing of the government after the shock (the government continues to access the credit market afterwards). However, the large capital losses reduce credit supply.

   Because the losses are born by the banks, household feel now wealthier. This leads to a reduction in labor supply and reinforces the fall in output. (It works opposite to an increase in G, when hh feel poorer and work more)