International Macroeconomics & Finance - ECON 641

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Course overview

We will discuss broad topics

- Solving DSGE models, with particular attention to their nonlinearities
- Formulating DSGE models with nominal rigidities
- Formulating (and estimating) DSGE models with financial frictions, with particular attention to debt and housing
- Studying optimal policy in models with financial frictions

DSGE Models: The Simplest Example

RBC Model with zero Capital Depreciation and Fixed Labor Supply
The planner's problem can be written as:

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right)$$

subject to

$$K_t - K_{t-1} = A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t \tag{1}$$

Optimal consumption implies

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha \left(\frac{A_{t+1}}{K_t} \right)^{1-\alpha} + 1 \right) \right) \tag{2}$$

Assume technology follows an AR(1) process in logs, that is

$$\log A_t = \rho \log A_{t-1} + \log u_t \tag{3}$$

where ρ is the autocorrelation of the shock. $\log u$ has mean zero, finite variance.

Solution with Linearization

The system made by (1) to (3) is a non-linear system with rational expectations. We usually solve them in the following steps

- Find the steady state.

$$
\begin{aligned}
\log A &= 0- > A = 1 \\
C &= K^{\alpha} \\
1-\beta &= \alpha\beta\left(\frac{1}{K}\right)^{1-\alpha} - > \left(\frac{\alpha\beta}{1-\beta}\right)^{\frac{1}{1-\alpha}} = K
\end{aligned}
$$

- Linearize model equations around the steady state, letting $x_t \equiv \frac{X_t - X_{SS}}{X_{SS}}$

equation 1

$$C_t = A_t^{1-\alpha} K_{t-1}^{\alpha} - K_t + K_{t-1}$$

$$\begin{aligned}
\log C_t &= \log\left(A_t^{1-\alpha} K_{t-1}^{\alpha} - K_t + K_{t-1}\right) \\
&\quad \text{take total differential around steady state} \\
\frac{1}{C} dC_t &= \frac{1}{C}\left((1-\alpha) A^{-\alpha} K^{\alpha} dA_t + \alpha A^{1-\alpha} K^{\alpha-1} dK_{t-1} - dK_t + dK_{t-1}\right) \\
c_t &= (1-\alpha) A_t + \alpha k_{t-1} - \frac{K}{C} k_t + \frac{K}{C} k_{t-1}
\end{aligned}$$

- equation 2

$$
\begin{aligned}
E_t\left(\frac{C_{t+1}}{C_t}\right) &= \beta E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha}+1\right) \\
& \text{steady state of both sides is 1} \\
E_t c_{t+1} - c_t &= \alpha\left(1-\alpha\right)\left(\frac{1}{K}\right)^{1-\alpha}\left(E_t a_{t+1} - k_t\right) \\
0 &= -E_t c_{t+1} + c_t + \frac{\left(1-\alpha\right)\left(1-\beta\right)}{\beta}\left(E_t a_{t+1} - k_t\right)
\end{aligned}
$$

- equation 3

$$a_t = \rho a_{t-1} + u_t$$

Taking Stock

This is a dynamic system of 3 equations in 3 unknowns. To use a more compact notation, we prefer to write it in the following form

$$
\begin{aligned}
0 &= E_t\left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t\right] \quad (4) \\
\mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t \quad (5)
\end{aligned}
$$

where

- $\mathbf{x}_t$ is the vector collecting all the endogenous variables of the model.
- $\mathbf{z}_t$ collects all the exogenous stochastic processes.

In our above example

$$\mathbf{x} = \begin{bmatrix} c \\ k \end{bmatrix}, \mathbf{z} = [a]$$

and $\mathbf{F} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} -1 & -\frac{\alpha\beta}{1-\beta} \\ 1 & -\frac{(1-\alpha)(1-\beta)}{\beta} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 0 & \frac{\alpha}{1-\beta} \\ 0 & 0 \end{bmatrix},$

$\mathbf{L} = \begin{bmatrix} 0 \\ \frac{(1-\alpha)(1-\beta)}{\beta} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix}, \mathbf{N} = [\rho]$

To summarize, a linearized DSGE model can be written in the following form

$$
\begin{aligned}
0 &= E_t\left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{u}_{t+1} + \mathbf{M}\mathbf{u}_t\right] \\
\mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t
\end{aligned}
$$

The recursive equilibrium law of motion describes endogenous variables as function of the state:

$$
\mathbf{x}_t = \mathbf{P}\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{z}_t \tag{6}
$$

i.e., matrices **P**, **Q** such that the equilibrium is described by these rules.

- Finally, what we do is to plug the matrices in (4) and (5) in a computer, to obtain (6) .

In our toy example above, set $\alpha = 0.33, \beta = 0.99, \rho = 0.98$. Then (from `runrbc.m`)

$$
\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} 0 & 0.6589 \\ 0 & 0.9899 \end{bmatrix} \begin{bmatrix} c_{t-1} \\ k_{t-1} \end{bmatrix} + \begin{bmatrix} 0.1755 \\ 0.0151 \end{bmatrix} [z_t]
$$

Practice.

Let problem for planner be: $\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \left(\log C_s + \tau \log \left(1 - L_t\right)\right)\right)$
subject to $K_t - K_{t-1} = A_t^{1-\alpha} K_{t-1}^{\alpha} L_t^{1-\alpha} - C_t$
let $\tau = 1$.

1. Derive planner's first order conditions.
2. Find analytical steady state.
3. Solve for decision rules of the form $\mathbf{x}_t = \mathbf{P}\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{z}_t$.
4. Compare coefficients of P and Q with the ones obtained in the model with fixed labor supply.

Solving Models with Occasionally Binding Constraints

Occasionally binding constraints arise in many economic applications. Examples include:
Models with limitations on the mobility of factors of production;

Models with heterogenous agents and constraints on the financial assets available to agents;

Models with a zero lower bound on the nominal interest rate;

Models with inventory management.

Why is a Toolkit Needed?

Encompassing realistic features to improve model fit in empirically driven applications may quickly raise the number of state variables. This may render standard global solution methods, such as dynamic programming, infeasible.

An alternative that has been used in practice, especially in applications that deal with the zero lower bound on policy rates, is to use a piece-wise perturbation approach.

This approach has the distinct advantage of delivering a solution for models with a large number of state variables. Furthermore, it can be easily extended to encompass multiple occasionally binding constraints.

1) Present a toolbox that extends Dynare to use this solution technique.
2) Can gauge performance of this approach relative to other solution methods (more accurate but slower, for instance).

The Solution Method

- The linearized system of necessary conditions for an equilibrium of a baseline DSGE model can be expressed as:

$$\mathcal{A}_1 E_t X_{t+1} + \mathcal{A}_0 X_t + \mathcal{A}_{-1} X_{t-1} + \mathcal{B} u_t = 0. \tag{M1}$$

  Where X_t are variables in deviation from non-stochastic steady state. There are situations however (away from ss) when one (or more) of the equilibrium conditions may not hold, and is replaced by another one. When the "starred" system applies, express the system as:

$$\mathcal{A}_1^* E_t X_{t+1} + \mathcal{A}_0^* X_t + \mathcal{A}_{-1}^* X_{t-1} + \mathcal{B}^* u_t + \mathcal{C}^* = 0 \tag{M2}$$

  where C^* is a vector of constants.

- Both systems are linearized around the same point – same X across systems.

The Solution Method

- When the baseline model applies (M1), we use standard methods to express solution as:

$$X_t = \mathcal{P}X_{t-1} + \mathcal{Q}u_t. \qquad \text{(M1\_DR)}$$

- If the starred model applies (M2), *shoot back* towards the initial condition starting from the last period before we return to M1.
- Main idea: suppose that M2 applies in t, but M1 is expected to apply in all future periods $t+1$, the decision rule in t is:

$$\underset{\text{M2}}{\mathcal{A}_1^*} \; \underset{\text{M1\_DR}}{\mathcal{P}X_t} + \mathcal{A}_0^* X_t + \underset{\text{M2}}{\mathcal{A}_{-1}^*} X_{t-1} + \mathcal{C}^* = 0,$$

$$X_t = -\left(\mathcal{A}_1^*\mathcal{P} + \mathcal{A}_0^*\right)^{-1}\left(\mathcal{A}_{-1}^* X_{t-1} + \mathcal{B}^* u_t + \mathcal{C}^*\right)$$

- One can proceed in a similar fashion to construct the time-varying decision rules when M2 applies for multiple periods.
- In each period in which M2 applies, the expectation of how long one expects to stay in M2 affects the value of X_t today

The Solution Method

The search for the appropriate time-varying decision rules implies that for each set of shocks at a point in time one needs to calculate the expected future duration of each "regime."

Truncate the simulation at an arbitrary point and reject the truncation if the solution implies that the model has not returned to the reference regime by that point.

Start with a guess of the expected durations that is based on the linear solution. Update the guess based on where the conditions of system 1 are violated using the piece-wise linear method until no violation remains.

An Example

- To fix ideas, let's first consider a simple, forward-looking, linear model:

$$\begin{aligned} q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\ r_t &= \max\left(\underline{r}, \phi q_t\right) \end{aligned}$$

  where u_t is an *iid* shock.
- The general solution for q_t takes the form

$$\begin{aligned} q_t &= \varepsilon_{qq,t} q_{t-1} + \varepsilon_{qu,t} u_t + c_{q,t} \\ r_t &= \varepsilon_{rr,t} q_{t-1} + \varepsilon_{ru,t} u_t + c_{r,t} \end{aligned}$$

- In turn, the ε are functions of q_{t-1} and u_t.
- How do we find the solution?

An Example: Solution ignoring the constraint

Ignore the constraint first

$$\begin{aligned} q_t &= \frac{\beta(1-\rho)}{1+\sigma\phi} E_t q_{t+1} + \frac{\rho}{1+\sigma\phi} q_{t-1} + \frac{1}{1+\sigma\phi} u_t \\ q_t &= a E_t q_{t+1} + b q_{t-1} + c u_t \end{aligned}$$

Find solution (method of undetermined coefficients)

$$\begin{aligned} q_t &= \varepsilon_q q_{t-1} + \varepsilon_u u_t \text{ (guess)} \\ E_t q_{t+1} &= \varepsilon_q q_t \text{ (expectation given guess)} \\ a E_t q_{t+1} &= a \varepsilon_q q_t = a \varepsilon_q^2 q_{t-1} + a \varepsilon_q \varepsilon_u u_t \end{aligned}$$

Match coefficients

$$\underbrace{e q_{t-1} + \varepsilon_u u_t}_{q_t} = \underbrace{a \varepsilon_q^2 q_{t-1} + a \varepsilon_q \varepsilon_u u_t}_{a E_t q_{t+1}} + b q_{t-1} + c u_t$$

so that (after picking the "stable" root)

$$\varepsilon_q = a \varepsilon_q^2 + b, \; \varepsilon_u = a \varepsilon_q \varepsilon_u + c$$

$$\varepsilon_q = \left(1 - \sqrt{1-4ab}\right)/2a, \varepsilon_u = c/\left(1 - a\varepsilon_q\right)$$

Plug some numbers

$$\begin{aligned}\beta &= 0.99\\ \phi &= 1\\ \rho &= 0.5\\ \sigma &= 1\\ \underline{r} &= -0.02\end{aligned}$$

In this case (see `runsimmodelsimple.m`)

$$\begin{aligned}\varepsilon_q &= 0.2677\\ \varepsilon_u &= 0.5355\end{aligned}$$

so that

$$q_t = r_t = 0.2677q_{t-1} + 0.5355u_t$$

Is this solution always correct? Consider the case of a large negative shock to u in period 1. If $q_0 = 0$, any u_t such that

$$\begin{aligned} r^* &= 0.5355u^* < -0.02 \\ u^* &< -0.0373 \end{aligned}$$

will violate constraint.
Suppose for instance $u_1 = -0.2$. Ignoring constraint, solution is

$$r_t = 0.2677r_{t-1} + 0.5355u_t$$

$$\begin{aligned} r_1 &= -0.5355 * 0.2 = -0.1071 \\ r_2 &= 0.2677r_1 = -0.0287 \\ r_3 &> -0.02 \end{aligned}$$

Hence ignoring the constraint r_t would be below -0.02 for 2 periods. Moreover, there is a feedback loop. Higher values of r imply lower q, which implies lower desired values of r, so r can end up being at $\underline{r}$ for longer

We use a guess and verify method to determine how long the constraint will bind. We start by guessing durations that are based on the linear solution that ignores the constraint. Iterate until convergence.

So the first guess is going to be 2 periods.

–> Suppose we guess that r remains at ϕ for $t\_low = 2$ periods.
–> Because r is not going to be low as guessed in linear solution ....
–> q will fall more than if r did not hit the constraint ...
–> and r might in turn stay at its lowest bound ϕ more than t_low periods.

In all interesting cases, first guess is not last guess, since dynamics of system depend on feedback loop between duration of constraint and endogenous reaction of variables to constraint. In the example above, one can think of a New Keynesian model at the ZLB.

Now cast system using our general notation (use $\beta' = \beta(1-\rho)$):

$$\begin{aligned} q_t &= \beta' E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\ r_t &= \max(\underline{r}, \phi q_t) \end{aligned}$$

$$\mathcal{A}_1 E_t X_{t+1} + \mathcal{A}_0 X_t + \mathcal{A}_{-1} X_{t-1} + \mathcal{B} u_t = 0. \tag{M1}$$

$$\begin{bmatrix} -\beta' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ r_1 \end{bmatrix} + \begin{bmatrix} 1 & \sigma \\ -\phi & 1 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix} + \begin{bmatrix} -\rho & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ql \\ rl \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\mathcal{A}_1^* E_t X_{t+1} + \mathcal{A}_0^* X_t + \mathcal{A}_{-1}^* X_{t-1} + \mathcal{B}^* u_t = -\mathcal{C}^* \tag{M2}$$

$$\begin{bmatrix} -\beta' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ r_1 \end{bmatrix} + \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix} + \begin{bmatrix} -\rho & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ql \\ rl \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ \bar{r} \end{bmatrix}$$

and

$$X_t = \mathcal{P} X_{t-1} + \mathcal{Q} u_t \tag{M1\_DR}$$

$$\begin{bmatrix} q_t \\ r_t \end{bmatrix} = \begin{bmatrix} \varepsilon_q & 0 \\ \varepsilon_q & 0 \end{bmatrix} \begin{bmatrix} q_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_u \\ \varepsilon_u \end{bmatrix} u$$

We guess that in t=3 normal system applies. Hence need to find solution in t=1, 2, given the shock taking place in period 1, and knowing X_0. In that case, the solution in t=2 should satisfy

$$\begin{aligned} X_2 &= -\left(\mathcal{A}_1^*\mathcal{P} + \mathcal{A}_0^*\right)^{-1}\left(\mathcal{A}_{-1}^* X_1 + \mathcal{B}^* u_2 + \mathcal{C}^*\right) \\ &= P_2 X_1 + Q_2 u_2 + C_2 \end{aligned}$$

where

$$\begin{aligned} P_2 &= -\left(\mathcal{A}_1^*\mathcal{P} + \mathcal{A}_0^*\right)^{-1}\mathcal{A}_{-1}^*, \\ Q_2 &= -\left(\mathcal{A}_1^*\mathcal{P} + \mathcal{A}_0^*\right)^{-1}\mathcal{B}^* u_2, \quad C_2 = -\left(\mathcal{A}_1^*\mathcal{P} + \mathcal{A}_0^*\right)^{-1} C^* \end{aligned}$$

I plug the numbers now. Using

$$\begin{aligned} \mathcal{A}_1^* &= \begin{bmatrix} -0.495 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{P} = \begin{bmatrix} 0.2677 & 0 \\ 0.2677 & 0 \end{bmatrix}, \mathcal{A}_0^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathcal{B}^* &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathcal{C}^* = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix} \end{aligned}$$

I get

$$X_2 = \underbrace{\begin{bmatrix} 0.5706 & 0 \\ 0 & 0 \end{bmatrix}}_{P_2} X_1 + \underbrace{\begin{bmatrix} .023 \\ -0.02 \end{bmatrix}}_{C_2}$$

We do not know yet X_1 the solution in period 1. Assuming $(M2)$ applies in $t=1$ and is expected to apply in $t=2$, the solution in 1 is

$$\mathcal{A}_1^*\left(\mathcal{P}_2 X_1 + C_2\right) + \mathcal{A}_0^* X_1 + A_{-1}^* X_0 + B u_1 + \mathcal{C}^* = 0,$$
$$X_1 = -\left(\mathcal{A}_1^* \mathcal{P}_2 + \mathcal{A}_0^*\right)^{-1}\left(\mathcal{B}^* u_1 + \mathcal{C}^* + A_1^* C_2 + A_{-1}^* X_0\right)$$
$$X_1 = P_1 X_0 + Q_1 u_1 + C_1$$

where

$$\begin{aligned}
\mathcal{P}_1 &= -\left(\mathcal{A}_1^* \mathcal{P}_2 + \mathcal{A}_0^*\right)^{-1} A_{-1}^* \\
Q_1 &= -\left(\mathcal{A}_1^* \mathcal{P}_2 + \mathcal{A}_0^*\right)^{-1} \mathcal{B}^* \\
\mathcal{C}_1 &= -\left(\mathcal{A}_1^* \mathcal{P}_2 + \mathcal{A}_0^*\right)^{-1} \left(\mathcal{C}^* + A_1^* \mathcal{C}_2\right)
\end{aligned}$$

After plugging in all the numbers, assuming $X_0 = 0$, we get

$$X_1 = \begin{bmatrix} -0.235\,00 \\ -0.02 \end{bmatrix}$$

So far we went backwards from the last period in which regime 2 applies to the first.
Now we go forward. Plug X_1 back into solution for X_2 and get

$$X_2 = \begin{bmatrix} 0.57064 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.23500 \\ -0.02 \end{bmatrix} + \begin{bmatrix} .023 \\ -0.02 \end{bmatrix} = \begin{bmatrix} -0.1111 \\ -0.02 \end{bmatrix}$$

and now plug X_2 into X_3

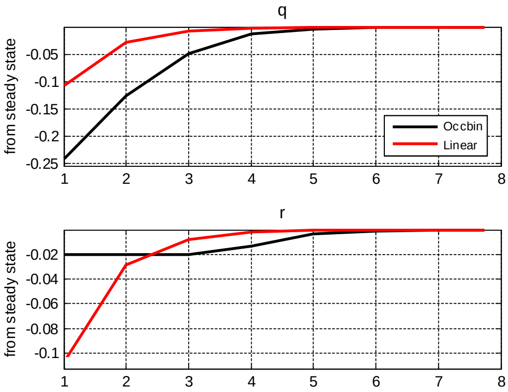
$$\begin{aligned} X_3 &= \mathcal{P} X_2 \\ X_3 &= \begin{bmatrix} -0.0297 \\ -0.0297 \end{bmatrix} \end{aligned}$$

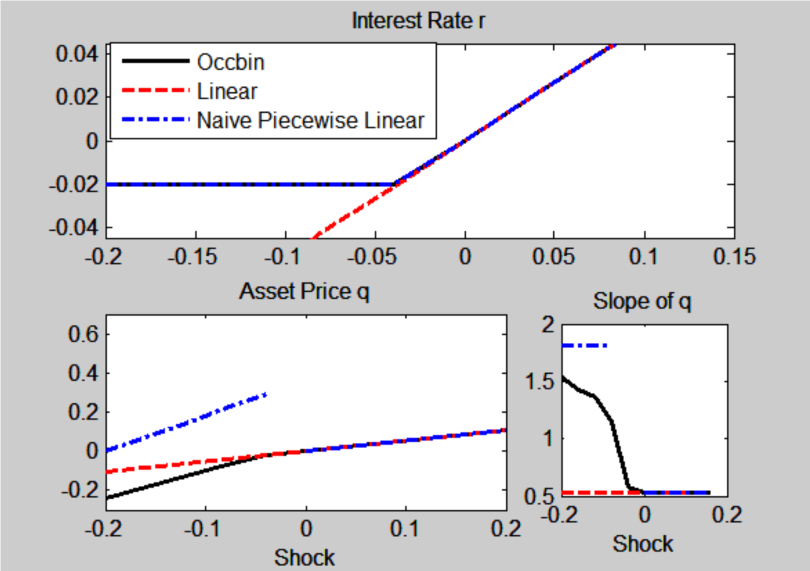
which violates constraint in 3.

- Note the need to update guess.
- We guessed that starred system $(M2)$ applies in 1 and 2 and that the normal applies in 3. Based on this guess, the starred system applies in 3.
- Hence we update the guess that starred system applies for 3 periods.
- Redo the whole thing again until the guessed duration in the starred regime coincides with the actual duration.

In the next step, we assume that the normal system applies in 4 but the starred applies in 1, 2 and 3, solve for P_3, Q_3 and C_3, use them to compute X_2 and X_1, and go back to see if X_3 satisfies the constraints (it does, I have checked it myself)

Impulse Responses to a $u_1 = -0.2$ shock





Interest Rate r
Occbin
Linear
Naive Piecewise Linear
0.04
0.02
0
-0.02
-0.04
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
Asset Price q
0.6
0.4
0.2
0
-0.2
-0.2
-0.1
0
0.1
0.2
Shock
Slope of q
2
1.5
1
0.5
-0.2
0
0.2
Shock

Example 1: Borrowing Constraint Model

- To check if method is accurate, we apply it to models for which we can compute a a full non-linear solution to arbitrary precision using dynamic programming methods.
- Consider simple model where a random endowment y_t that can be used as collateral

$$u = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log(c_t) \right\}$$

$$c_t = y_t + b_t - 1.05 b_{t-1}$$

$$b_t \leq 2 y_t \tag{c1}$$

$$\log(y_t) = \rho \log(y_{t-1}) + \sigma \sqrt{1-\rho^2} \varepsilon_t$$

$$\varepsilon_t \tilde{} N(0,1), \ \sigma = 0.03, \ \rho = 0.9$$

- We look at how solution method handles cases when increases in y_t are large enough so that constraint is not binding. We try $\beta = 0.94$ and $\beta = 0.949$
- Here: constraint $(c1)$ BINDS in *normal* times.

Example 2: RBC with Irreversible Capital

- Investment cannot fall below a given threshold

$$
\begin{gathered}
u = E_0 \left\{ \sum_{t=0}^{\infty} 0.96^t \log\left(c_t\right) \right\} \\
c_t + k_t - 0.9 k_{t-1} = A_t k_{t-1}^{0.33} \\
k_t - 0.9 k_{t-1} \geq \phi k_{t-1} \qquad \text{(c2)} \\
\log\left(A_t\right) = 0.9 \log\left(A_{t-1}\right) + \sigma \sqrt{1-\rho^2} \varepsilon_t \\
\varepsilon_t \tilde{} N\left(0,1\right),\ \sigma = 0.03,\ \rho = 0.9
\end{gathered}
$$

where $\phi > 0$.

- Here: constraint $(c2)$ DOES NOT bind in *normal* times.

Example 3: Borrowing and Housing

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t + j \log h_t\right)$$

$$\begin{aligned} c_t + q_t h_t &= y + b_t - R b_{t-1} + q_t h_{t-1} \left(1 - \delta\right) \\ b_t &\leq m q_t h_t \\ \log q_t &= \rho \log q_{t-1} + v_t \end{aligned}$$

Here the FOCs would be

$$\mu_t \left(b_t - m q_t h_t\right) = 0$$
$$u' \left(c_t\right) = \beta R E_t u' \left(c_{t+1}\right) + \mu_t$$
$$q_t u' \left(c_t\right) = u' \left(h_t\right) + \beta \left(1 - \delta\right) E_t q_{t+1} u' \left(c_{t+1}\right) + \mu_t m q_t$$

Assuming $\beta R < 1$, here the borrowing constraint binds in normal times.

Structure of Solution Programs (Dynare)

The programs we devised take as input two Dynare model files.
One .mod file specifies the normal M1 model from which we calculate

$$A_1 E_t X_{t+1} + A_0 X_t + A_{-1} X_{t-1} = 0.$$

The other .mod file specifies the starred model M2 with the occasionally binding constraint inverted (binding if it was not binding in the reference model, or not binding if it was binding in the reference model). This .mod file yields

$$A_1^* E_t X_{t+1} + A_0^* X_t + A_{-1}^* X_{t-1} + C^* = 0.$$

We use the analytical derivatives computed by Dynare to construct $A_1, A_0, A_{-1}, A_1^*, A_0^*, A_{-1}^*$, and C^*.

M1: hp.mod

y=1;
c+q*h=y+b-R*b(-1)+q*h(-1)*(1-δ);
b=M*q*h;
lb=1/c-β*R/c(+1);
q/c=j/h+β*(1-δ)*q(+1)/c(+1)+lb*M*q;
lev=b/(M*q*h)-1;
log(q)=ρ*log(q(-1))+u;

M2: hpnotbinding.mod

y=1;
c+q*h=y+b-R*b(-1)+q*h(-1)*(1-δ);
lb=0;
lb=1/c-β*R/c(+1));
q/c=j/h+β*(1-δ)*q(+1)/c(+1)+lb*M*q;
lev=b/(M*q*h)-1;
log(q)=ρ*log(q(-1))+u;

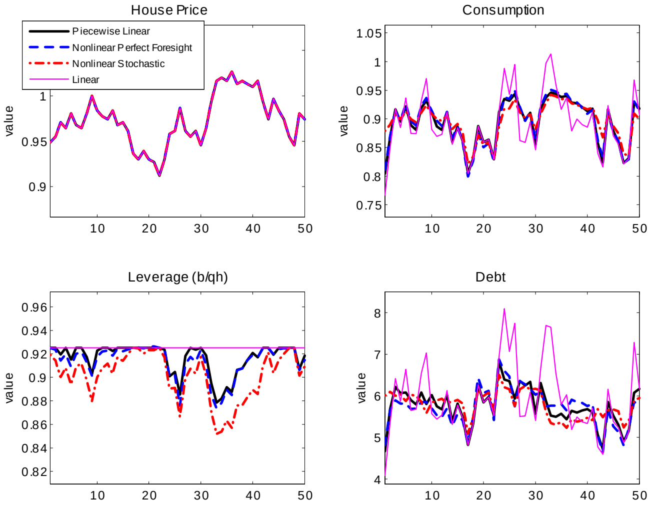
The main file `runsim_hp.m` contains

1. mod files: `model1 = 'hp'; model2 = 'hpnotbinding';`
2. constraint violation triggers switch to m2: `constraint='lb<-lb_ss';`
3. constraint violation triggers switch to m1: `constraint_relax1='lev>0'`
4. solve model1 to obtain M1_DR; write model2 in M2 form
5. given shocks, check if model1 assumptions are violated, if so, look for solution

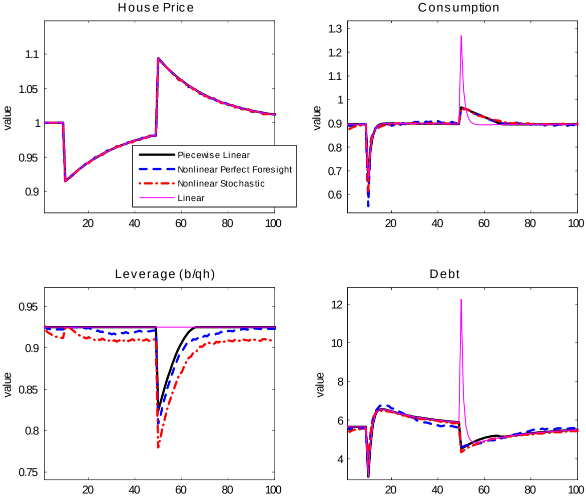
function [zdata zdataconcatenated oobase_ Mbase_] = ...
solve_one_constraint(model1,model2,...
constraint, constraint_relax,...
shockssequence,irfshock,nperiods,maxiter)

- `model1, model2`: dynare mod files containing (linear or nonlinear) model equations
- `constraint, constraint_relax`: strings with constraints that have to be verified
  constraint defines the first constraint
  if constraint is true, solution switches to model2
  but if constraint_relax is true, solution reverts to model1
- `shockssequence`: `sequence of innovations under which one wants to solve model`
  e.g. `randn(100,1)*`σ_j for simulation
  or `[ 1; zeros(50,1) ]` for impulse responses

Simulations - Borrowing and Housing Model



IRF - Borrowing and Housing Model



Accuracy – Borrowing and Housing Model

	Log Consumption		Correlations		$\frac{b}{qh}$	ΔWelf.
	st.dv	skewness	$\ln q, \ln c$	$\ln q, \frac{b}{qh}$	mean	
Linear	6.1%	0.03	0.40	0.00	0.925	0.18%
Occbin	4.5%	-1.17	0.54	-0.60	0.911	0.02%
Nonlin.perf.fores.	4.6%	-1.20	0.53	-0.58	0.910	0.01%
Nonlinear stoch.	3.7%	-1.30	0.65	-0.71	0.896	—

Nonlinear models

Nonlinear structural models can be described by:

1. The model's equilibrium conditions and FOCs

$$\Gamma\left(E_t z_{t+1}, z_t, z_{t-1}, \varepsilon_t\right) = 0$$

   where z_t includes s_t and c_t, "states" and "controls".
   $E_t z_{t+1}$ is an unknown object!

2. The solution is a set of policy functions ζ

$$z_t = \zeta\left(z_{t-1}, \varepsilon_t\right)$$

   such that, for any value of $\left(z_{t-1}, \varepsilon_t\right)$

$$F\left(z_{t-1}, \varepsilon_t\right) \equiv \Gamma\left(E_t \zeta\left(z_t, \varepsilon_{t+1}\right), \zeta\left(z_{t-1}, \varepsilon_t\right), z_{t-1}, \varepsilon_t\right) = 0$$

One Example

- Example of DSGE model we want to solve:

$$C_t^{-\gamma} = \beta \delta_t R_t E_t \left(C_{t+1}^{-\gamma} / \Pi_{t+1} \right) \tag{1}$$

$$w_t = N_t^{\eta} C_t^{\gamma} \tag{2}$$

$$\frac{N_t}{C_t^{\gamma}} \left(\psi \left(\Pi_t - 1 \right) \Pi_t - (1-\theta) - \theta w_t \right) = \beta \delta_t E_t \left(\frac{N_{t+1}}{C_{t+1}^{\gamma}} \psi \left(\Pi_{t+1} - 1 \right) \Pi_{t+1} \right) \tag{3}$$

$$N_t = C_t + \frac{\psi}{2} \left(\Pi_t - 1 \right)^2 N_t \tag{4}$$

$$R_t = \max \left(1, \Pi_t^{\phi} / \beta \right) \tag{5}$$

- Solution is a set of policy functions $C_t = C\left(\delta_t\right), R_t = R\left(\delta_t\right), w_t = w\left(\delta_t\right), \Pi_t = \Pi\left(\delta_t\right), N_t = N\left(\delta_t\right)$ such that (1) to (5) hold for every value of δ_t.

A Simpler Example

- Consider model given by:

$$p_t = \beta E_t \left(\frac{d_{t+1}^{-\gamma}}{d_t^{-\gamma}} \left(p_{t+1} + d_{t+1} \right) \right)$$
$$d_t = \exp\left(\rho \log d_{t-1} + e_t \right)$$

- Two equations, two unknowns.
- Can reinterpret solving the model as solving for *unknown expectation function* $\phi_t = \beta E_t \left(\frac{d_{t+1}^{-\gamma}}{d_t^{-\gamma}} \left(p_{t+1} + d_{t+1} \right) \right)$
- Parameterized expectations are one way of solving such nonlinear systems
- The trick: use model simulations to compute the expectation function using simulated data

Nonlinear techniques: Parameterized Expectations

- The idea: *Approximate* conditional expectation in forward-looking equations by *parameterized* functions of the state variables.
- Why? Because the true conditional expectation is only a function of information available at time t
- If we can find a "good" function capturing the expectation, then any other piece of information should be orthogonal to the function itself.
- We can find this function by guessing/positing its form and its explanatory variables, and then verifying ex post that such function is a “good” one
- Choose functions to maximize fit

A Formal Statement of PEA

- The underlying model to solve is

$$F\left(E_t\left(G\left(p_{t+1}, p_t, d_{t+1}, d_t\right), d_t, p_t, \varepsilon_t\right)\right) = 0 \qquad \text{(DSGE model)}$$

- PEA replaces the unknown $G\left(p_{t+1}, p_t, d_{t+1}, d_t\right)$ functions with **parametric** functions of the form $\phi\left(d_t; \boldsymbol{\theta}\right)$

$$\text{replace } E_t\left(G\left(p_{t+1}, p_t, d_{t+1}, d_t\right)\right) \text{ with } \phi\left(d_t; \boldsymbol{\theta}\right) \qquad \text{(parametric function)}$$

- The approximated model then becomes

$$F\left(E_t\left(\phi\left(d_t; \boldsymbol{\theta}\right), d_t, p_t, \varepsilon_t\right)\right) = 0 \qquad \text{(Approx DSGE model)}$$

- PEA algorithm finds θ such that

$$\theta = \arg\min \left\|\phi\left(d_t; \boldsymbol{\theta}\right) - E_t\left(G\left(p_{t+1}, p_t, d_{t+1}, d_t\right)\right)\right\|^2 \qquad (\theta)$$

so θ satisfies rational expectations

Steps of PEA

- 1. Initialization. Generate long time series of exogenous shocks d_t for $t = 1 : T$
- 2. Make initial guess of the function parameterizing expectations, call $\phi_0 \left(d_t ; \boldsymbol{\theta} \right)$
- 3. For $t = 1 : T$ generate series of all variables $\{d_t, p_t\}_{t=1}^{T}$ and all realizations of $\{G_t\}_{t=1}^{T}$ based on the initial guess.
  (In complicated system, this might require solving in each period a system of nonlinear equations. Can use fsolve.m or csolve.m to do that)

$$\begin{aligned} d_t &= \exp\left(\rho \log d_{t-1} + e_t\right) \\ p_t &= \phi_0\left(d_t\right) \\ G_t &= \beta \frac{d_{t+1}^{-\gamma}}{d_t^{-\gamma}} \left(p_{t+1} + d_{t+1}\right) \end{aligned}$$

(note: G is what is "inside" the expectation function, not the expectation; ϕ is the function that wishes to approximate the expectation of G)

Steps of PEA (continued)

- 4. Find $\boldsymbol{\theta}$ that minimizes expectation error. That is, do a regression with (1) $G_t = \beta \frac{d_{t+1}^{-\gamma}}{d_t^{-\gamma}} \left(p_{t+1} + d_{t+1}\right)$ as dependent variable; (2) d_t as RHS variable, or $\phi\left(d\right)$ as function; (3) θ as parameters to be estimated
- 5. Continue until $\boldsymbol{\theta}_{n+1}$ and $\boldsymbol{\theta}_n$ (or until $\left(\{p_t, d_t\}_{t=1}^T\right)_{n+1}$ and $\left(\{p_t, d_t\}_{t=1}^T\right)_n$) are close.

[When G's and $\boldsymbol{\theta}$ are close, the solution approximates rational expectations. (see Marcet and Marshall (1992) for a proof)]

Example

- Set: $\gamma = 1, \beta = 0.95, \rho = 0.9, \sigma = 0.01$.
- Guess $\phi_t^0 = \beta / (1-\beta) + 0 \times d_t$ (that is, $\boldsymbol{\theta}_0 = \begin{bmatrix} \beta/(1-\beta) & 0 \end{bmatrix}'$)
- Given realization of dividends, in first iteration $n = 1$ we will compute, in this order,

	exo	ϕ	solve	compute G		
t	d_t	ϕ_0	$p_t = \phi_0$	p_{t+1}	d_{t+1}	$G_t = \frac{\beta d_{+1}^{-\gamma}(p_{+1}+d_{+1})}{d_t^{-\gamma}}$
1	1	19	19	19	1.0029	18.9483
2	1.0029	19	19	19	1.0095	18.8835
3	1.0095	19	19	19	0.9942	19.2876
4	0.9942	19	19	19	0.971	19.4262
...						...

- Regress G_t on constant and d_t; get a new ϕ which will "update" the old ϕ_0

$$\begin{aligned} G &= 19.0002 + 2.64\,(d-1) + error\_term \\ &\longrightarrow \phi_{new} = E\,(G) = 19.0007 + 2.62\,(d-1) \end{aligned}$$

- Update coefficients $\boldsymbol{\theta}$ of the ϕ function according to some dampening scheme

$$\boldsymbol{\theta}^1 = \alpha \boldsymbol{\theta}^0 + (1-\alpha)\,\widehat{\boldsymbol{\theta}}_t^0, 0 < \alpha < 1$$

Example (continued)

- In the second step, $n = 2$, $(\alpha = 1/2)$

	exo	ϕ	solve	compute G		
t	d_t	ϕ_1	$p_t = \phi_1$	p_{t+1}	d_{t+1}	$G_t = \frac{\beta d_{+1}^{-\gamma}(p_{+1}+d_{+1})}{d_t^{-\gamma}}$
1	1	19.0002	19.0002	19.0004	1.0029	18.9483
2	1.0029	19.0004	19.0004	19.0129	1.0095	18.8835
3	1.0095	19.0129	19.0129	18.9926	0.9942	19.2876
4	0.9942	18.9926	18.9926	18.9619	0.971	19.4262
...						...

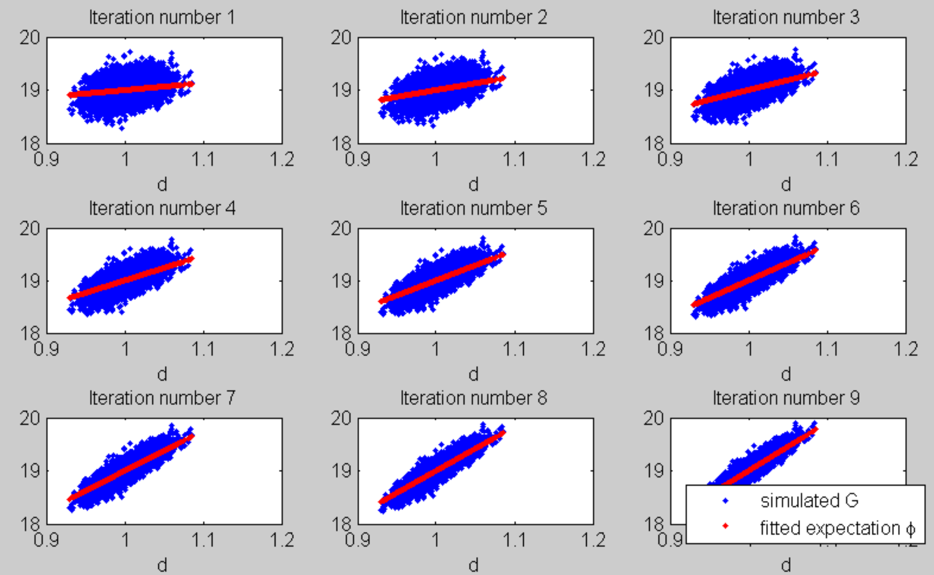
- Regress the new G_t on constant and d_t to get

$$\phi_{newer} = E(G) = 19.0007 + 3.78\,(d - 1)$$

- ... Keep doing this. After 100 iterations,

$$\begin{aligned} \theta_{100} &= \begin{bmatrix} 19 & 18.98 \end{bmatrix}' \\ \phi_{100} &= 19 + 18.98\,(d - 1) \end{aligned}$$

which "approximates" the rational expectation solution


Iteration number 1
Iteration number 2
Iteration number 3
Iteration number 4
Iteration number 5
Iteration number 6
Iteration number 7
Iteration number 8
Iteration number 9
20
19
18
0.9
1
1.1
1.2
d
simulated G
fitted expectation ϕ

PEA and Learning

- PEA may be viewed as a generalized method of undetermined coefficients, in which economic agents learn the decision rule at each step of the algorithm.
- Start with a guess of the decision rule
- Use outcomes to minimize expectation error (difference b/w actual and fitted) at each iteration step, until no further learning is possible

Practical Issues

- Marcet and Lorenzoni (1999) offer some suggestion to make the algorithm work in practice (optimal T, choice of function, to log or to not log, etc...)
- Typical problem: algorithm may not converge because of simulation error Judd, Maliar and Maliar (QE, 2011) suggestion: at each simulation step, replace actual G_t with its expectation calculated using quadrature-based methods. State space is constructed using stochastic simulation, expectations are computed using more accurate methods.

$$-\text{Given } p_t, d_t \text{ sequence we normally compute } G_t \text{ using}$$

$$G_t = \beta\left(\left(d_{t+1}/d_t\right)^{-\gamma}\left(p_{t+1}+d_{t+1}\right)\right)$$

– instead, given $p_t, d_t,$ replace G_t at each iteration using numerical integration

$$-\text{replace } e_t \text{ with a discretized version}$$

$$\text{e.g. } e_t \sim N(0,1) \text{ becomes } e_t = 1 \text{ wp } 1/2, \ -1 \text{ wp } 1/2$$

$$e_j = \begin{bmatrix} -1 & 1 \end{bmatrix}, \omega_j = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$G_t^* = \beta \sum_{j=1}^{J} \omega_j \left(\frac{\left(\rho d_t + e_{t,j}\right)^{-\gamma}}{d_t^{-\gamma}} \left(p_{t+1,j} + \left(\rho d_t + e_{t,j}\right)\right)\right)$$

Practice

Write a Matlab code that computes, using PEA, the solution to the stochastic growth model of chapter one. Compare the policy functions under PEA and linearization as best as you can – for instance, for $K_{t-1} = K_{SS}$, plot $K_t = f(A_t)$ under linearization and under PEA –. Use parameter values from Chapter 1.

$$
\begin{aligned}
K_t - K_{t-1} &= A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t \\
\frac{1}{C_t} &= \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha \left(\frac{A_{t+1}}{K_t} \right)^{1-\alpha} + 1 \right) \right) \\
\log A_t &= \rho \log A_{t-1} + \log u_t
\end{aligned}
$$

Money in Utility Model

- We present a basic model of money demand
- This model is the cornerstore to think about inflation, money, and the interaction between prices and economic activity
- Most macro models introduce money in one of the following ways
  - money yields direct utility or production services (Sidrauski 1967)
  - transaction costs of some form that give rise to a demand for money (Baumol-Tobin, Kiyotaki-Wright, CIA)
  - money is an asset to transfer resources intertemporally
- We consider the MIU model today
- Idea: real balances allow agents to save time in conducting their transactions: purchase of goods requires in other words the input of transaction services, and these transaction services are produced by money and time (see Walsh, Chapter 3, for more on this).

Money in Utility Model

- Households-businesses produce final good, supply labor, own capital, consume and invest, purchase one-period real bonds, hold money
- Government issues money, makes lump-sum transfers to the private sector with the real income generated by money creation.

agents/mkts	goods	money	bonds	*bc*
Household	$-c-K+f(K_{-1},L)+(1-\delta)K_{-1}+T$	$\frac{M_{-1}-M}{P}$	$R_{-1}B_{-1}-B$	$=0$
Govt	$-T$	$\frac{M-M_{-1}}{P}$		$=0$
equilibrium	$=0$	$=0$	$=0$	

Money in Utility Model

The representative household chooses $\{C_{t+i}, K_{t+i}, L_{t+i}, M_{t+i}/P_{t+i}\}_{i=0}^{\infty}$ to maximize:

$$\max_{C_t,L_t,K_t,M_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, \frac{M_t}{P_t}, L_t\right)$$

$$\text{with } u_C > 0,\ u_m > 0,\ u_L < 0$$

subject to:

$$C_t + K_t + \frac{M_t}{P_t} + B_t = R_{t-1}B_{t-1} + f\left(K_{t-1}, L_t\right) + (1-\delta)\,K_{t-1} + \frac{M_{t-1}}{P_t} + T_t$$

Money in Utility Model

- Why do we have real money $m_t \equiv M_t / P_t$ (as opposed to nominal) in the utility? What matters for agents' utility cannot be just the number of dollars that these individual hold, but rather how dollars can be exchanged for goods. Typically if you like good 1 and good 2, you write utility function as $u(x_1, x_2)$, rather than $u(x_1, p_2 x_2)$. Here idea is that services of money are proportional to how many goods you can buy with them allows to write $u = u(c_t, money\_services_t) = u(c_t, M_t / P_t)$
- We also assume that $du/dm > 0$: this implies that keeping consumption constant, an increase in m_t makes us happier. All in all, having m in the utility function is thus just a shortcut.
- Agents trade a real bond (claim on one unit of the consumption good) that offer a return in real terms R.
- In the competitive version, we can separate households and firms with households renting production factors to firms, and firm maximizing period by period profits solving a static problem. In this case, the RHS of the household constraint includes $r_t K_{t-1} + w_t L_t$ instead of $f(K_{t-1}, L_t)$

Optimality

From the household problem, we form the Lagrangean to find:

$$L = u\left(C_t, \frac{M_t}{P_t}, L_t\right) + \beta E_t u\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right) + ...$$
$$-\lambda_t\left(C_t + K_t + \frac{M_t}{P_t} + B_t - \left(R_{t-1}B_{t-1} + f\left(K_{t-1}, L_t\right) + (1-\delta)K_{t-1} + \frac{M_{t-1}}{P_t} + T_t\right)\right)$$
$$-\beta E_t \lambda_{t+1}\left([\cdots] - \left(R_t B_t + f\left(K_t, L_{t+1}\right) + (1-\delta)K_t + \frac{M_t}{P_t}\frac{P_t}{P_{t+1}} + T_{t+1}\right)\right) - ...$$

The first order conditions for this problem are:

$$u_{C,t} = \lambda_t \tag{C}$$
$$u_{m,t} = \lambda_t - \beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) \tag{m}$$
$$u_{Lt} = -\lambda_t f_{Lt} \tag{L}$$
$$\lambda_t = \beta E_t\left(R_t \lambda_{t+1}\right) \tag{B}$$
$$\lambda_t = \beta E_t\left(\left(1-\delta+f_{Kt}\right)\lambda_{t+1}\right) \tag{K}$$

And dropping the multiplier:

$$\begin{aligned} u_{C,t} &= \beta R_t E_t u_{C,t+1} && (1) \\ u_{C,t} &= \beta E_t \left(u_{C,t+1} \left(1 - \delta + f_{Kt} \right) \right) && (2) \\ 0 &= u_{L,t} + u_{C,t} f_{Lt} && (3) \\ u_{C,t} &= u_{m,t} + \beta E_t \left(u_{C,t+1} \frac{1}{\Pi_{t+1}} \right) && (4) \end{aligned}$$

Pricing Nominal Bonds

Combining 1+2 -> bonds offer in equilibrium same return as capital. Suppose you have nominal bonds Z_t traded offering a return I_t. The budget constraint is:

$$C_t + K_t + \frac{M_t}{P_t} + B_t + \frac{Z_t}{P_t} = I_{t-1}\frac{Z_{t-1}}{P_t} + R_{t-1}B_{t-1} + Y_t + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t$$

Optimality with respect to Z_t requires (

$$u_{C,t} = \beta I_t E_t \left(u_{C,t+1} / \Pi_{t+1} \right)$$

together with (1), $u_{C,t} = \beta R_t E_t u_{C,t+1}$, we get (if $cov\left(P_t/P_{t+1}, u_{C,t+1}\right) = 0$):

$$R_t = E_t \left(I_t / \Pi_{t+1} \right)$$

which is the Fisher parity, after Fisher. If the covariance is not zero,

$$\begin{aligned} R_t E_t u_{C,t+1} &= I_t E_t \left(\frac{P_t}{P_{t+1}} u_{C,t+1} \right) = I_t \left(E_t \left(\frac{P_t}{P_{t+1}} \right) E_t \left(u_{C,t+1} \right) + cov \left(\frac{P_t}{P_{t+1}}, u_{C,t+1} \right) \right) \\ R_t &= I_t E_t \left(\frac{P_t}{P_{t+1}} \right) + \frac{I_t}{E_t \left(u_{C,t+1} \right)} cov \left(\frac{P_t}{P_{t+1}}, u_{C,t+1} \right) \end{aligned}$$

$cov\left(P_t/P_{t+1}, u_{C,t+1}\right)$ defines the inflation risk premium

Pricing a Consol

A consol is a bond that pays one unit of consumption over forever. Suppose we want to price such an asset: in the budget constraint, a consol is an asset H whose ex-dividend price is q and that pays 1 every period from $t+1$ on:

$$C_t + A_t + q_t H_t = R_t A_{t-1} + (1 + q_t) H_{t-1}$$

Optimality with respect to H_t requires

$$u_{C,t} q_t = \beta E_t \left(u_{C,t+1} \left(1 + q_{t+1} \right) \right)$$

which implies

$$R_t = E_t \left(\frac{1 + q_{t+1}}{q_t} \right)$$

in log-linear terms, this expression can be rewritten as:

$$\widehat{q}_t = \beta E_t \widehat{q}_{t+1} - \widehat{R}_t$$

which shows how the price of a consol is negatively related to current and future short-term interest rates.

Pricing Money

The Euler condition for money is a typical expression for the price of an asset: if I give up consumption today and decide to hold money forever from then on, I will enjoy the stream of utility services in square brackets, which will be eroded from the rise in prices between t and the future.

$$\begin{aligned} u_{C,t} &= u_{m,t} + \beta E_t \left(u_{C,t+1} \frac{1}{\Pi_{t+1}} \right) \\ &= u_{m,t} + E_t \left(\frac{\beta}{\Pi_{t+1}} u_{m,t+1} + \frac{\beta^2}{\Pi_{t+1}\Pi_{t+2}} u_{m,t+2} + \frac{\beta^3}{\Pi_{t+1}\Pi_{t+2}\Pi_{t+3}} u_{C,t+3} \right) \end{aligned}$$

Money Demand

From the optimality condition for m and for nominal bonds

$$u_{C,t} = u_{m,t} + \beta E_t \left(u_{C,t+1} / \Pi_{t+1} \right) \text{ and } u_{C,t} = \beta E_t \left(\left(I_t / \Pi_{t+1} \right) u_{C,t+1} \right)$$

we obtain:

$$u_{C,t} = u_{m,t} + u_{C,t} / I_t$$

which implicitly defines a money demand function, so that we may write

$$\begin{aligned} u_{m,t} &\equiv g\left(m_t\right) = u_{C,t} - \frac{u_{C,t}}{I_t} \equiv h\left(C_t, I_t\right) \\ \frac{M_t}{P_t} &= \underset{g^{-1}h}{\varphi}\left(C_t, I_t\right) = \frac{\varphi\left(C_t, I_t\right)}{C_t} C_t \end{aligned}$$

rearranging, this can be written as:

$$\begin{aligned} M_t v_t &= P_t C_t \\ v_t &= \frac{C_t}{\varphi\left(C_t, I_t\right)} \end{aligned}$$

if changes in money supply do not affect C_t and I_t, there is a proportional relationship between money and prices.

Transversality Conditions

Together with this, we also have appropriate transversality conditions that state:

$$\lim_{t\to\infty} \beta^t u_{C,t} x_t = 0$$

for $x = B, K, m$.

An interpretation of the transversality condition is that it is not optimal to end up at infinity with keeping net assets if these are valuable. Loosely speaking, we impose the transversality condition by solving "forward" the forward-looking equations of the model.

Equilibrium

In each period the Government rebates to the public any real resources created by printing money:

$$T_t = \frac{M_t}{P_t} - \frac{M_{t-1}}{P_t}$$

To complete the description of the government problem, we also need to specify a money creation process (a rule to print money). A typical specification assumes that the growth rate of money supply follows a stationary process around a constant mean, that is:

$$M_t = \theta_t M_{t-1} \tag{5}$$

Below, we will assume that $\log \theta_t$ is random, has mean $\log \overline{\theta}$, and follows an AR(1) process around this mean. That is:

$$\ln \theta_t = (1-\rho)\ln \overline{\theta} + \rho \ln \theta_{t-1} + \varepsilon_t^M$$

where ε_t^M is zero mean, iid, with variance σ_ε^2. At quarterly frequency, a typical number for $\overline{\theta}$ would (used to) be 1.01.

Closing the Model

Once we have the optimality conditions, the only thing we need to close the model is the market clearing conditions. In equilibrium net supply of bonds is zero, since all agents are identical. Therefore:

$$C_t + K_t = Y_t + (1 - \delta) K_{t-1} \tag{6}$$

where

$$Y_t = A_t f(K_{t-1}, L_t) \tag{7}$$

and

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

We can then define a recursive equilibrium as follows:
An equilibrium is a sequence of values for $(C_t, K_t, L_t, m_t, \Pi_t, R_t, Y_t)$ *, given* $(K_{t-1}, m_{t-1}, A_{t-1}, \theta_{t-1})$ *and the sequence of monetary and technology shocks* $(\varepsilon_t^M, \varepsilon_t^A)$ *, satisfying at all times equations* (1) *to* (7) *as well as the transversality conditions for* K_t*,* B_t and m_t.

Functional Forms and Steady State

We want to consider the properties of this economy when it is in a steady state equilibrium in which nominal money grows at rate $\overline{\theta}$ and technology is constant. The steady state must satisfy conditions (1) to (7) above. That is:

$$\begin{aligned} C + \delta K &= Y \\ R &= \beta^{-1} \end{aligned}$$

moving to the monetary side, from $M_t = \theta_t M_{t-1}$, real balances M/P will be constant in steady state only if

$$\Pi = \overline{\theta}.$$

To move on, it is easier to assume some specific functional forms. Suppose we parametrize the utility as follows (here $m_t \equiv M_t/P_t$ denotes real money):

$$u_t = \frac{C_t^{1-\phi} m_t^{b(1-\phi)}}{1-\phi} - \frac{\tau L_t^{\eta}}{\eta}$$

ϕ is a parameter that dictates the separability between consumption and money balances. For the production function, we assume

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

In steady state, marginal utility of consumption, labor and money balances are:

$$\begin{aligned} u_C &= C^{-\phi} m^{b(1-\phi)} \\ u_L &= -\tau L^{\eta-1} \\ u_m &= b C^{1-\phi} m^{b(1-\phi)-1} \end{aligned}$$

Using the steady state version of (2) :

$$u_C = \beta u_C \left(1 - \delta + f_K(K)\right) \tag{2}$$

we obtain:

$$\begin{aligned} 1 &= \beta \left(1 - \delta + \frac{\alpha Y}{K}\right) \\ \frac{K}{Y} &= \frac{\alpha}{\beta^{-1} - (1-\delta)} \end{aligned}$$

Also, using $C = Y - \delta K$

$$\frac{C}{Y} = 1 - \frac{\delta K}{Y} = 1 - \frac{\alpha \delta}{\beta^{-1} - (1-\delta)} = c_0$$

On the monetary side:

$$I = R\Pi = \frac{\overline{\theta}}{\beta}$$

Finally, money holdings will be:

$$\begin{aligned}
\frac{u_m}{u_C} &= \frac{\overline{\theta} - \beta}{\overline{\theta}} = \frac{bC^{1-\phi} m^{b(1-\phi)-1}}{C^{-\phi} m^{b(1-\phi)}} \\
\frac{m}{C} &= b \frac{\overline{\theta}}{\overline{\theta} - \beta}
\end{aligned}$$

notice: in the data, $\frac{m}{C}$ has an economic interpretation.
The lower $\overline{\theta}$, the lower I, the higher m/C. Hence real money holdings depend negatively on inflation in this model. This intuition has generated lots of research into how big the welfare costs of inflation are. (the basic idea is that one can measure the welfare costs of inflation by the percentage increase in steady-state consumption necessary to compensate the decrease in utility associated with an increase in the steady state nominal interest rate.)

As for labor supply:

$$\begin{aligned}\tau L^{\eta-1} &= u_C f_L \\ \tau L^{\eta-1} &= C^{-\phi} m^{b(1-\phi)} (1-\alpha) \frac{Y}{L}\end{aligned}$$

Knowing that $C = c_0 Y$, we can solve for L as a function of C with some use of algebra.

$$\tau L^{\eta} = C^{-\phi} \left(b \frac{\overline{\theta}}{\overline{\theta} - \beta} C \right)^{b(1-\phi)} (1-\alpha) \frac{C}{c_0}$$

Summary of Steady State

For given parameters, we find steady state. For local dynamics we are interested in the ratios of variables relative to Y. These are the "big ratios". We treat ratios (m/C), (K/Y) and so on as observable to imply restrictions on the structural parameters that satisfy the big ratios (aka calibration).

$$
\begin{aligned}
\tau L^{\eta} &= \zeta C^{1-\phi+b(1-\phi)} \\
\frac{C}{Y} &= 1 - \frac{\alpha\delta}{\beta^{-1} - (1-\delta)} \\
\frac{m}{C} &= b\frac{\overline{\theta}}{\overline{\theta} - \beta} \\
\frac{K}{Y} &= \frac{\alpha}{\beta^{-1} - (1-\delta)} \\
Y &= AK^{\alpha}L^{1-\alpha}
\end{aligned}
$$

where

$$
\zeta = \left(b\frac{\overline{\theta}}{\overline{\theta} - \beta}\right)^{b(1-\phi)} \frac{(1-\alpha)}{c_0}
$$

Summary of Steady State (continued)

Note the following:

- In steady state, the ratios $\frac{K}{Y}$, $\frac{C}{Y}$ are independent of all parameters of the utility function and of the inflation rate.
- However, level of output may depend on inflation: the level of Y depends on L, which in steady state is a function of $\bar{\theta}$.
- Inflation in steady state just equals the growth rate of money supply.
- When $\phi = 1$, the model displays "superneutrality" of money in steady state: when $\phi = 1$, one can solve for the levels of C, L, Y and K independently of the steady state growth rate of money $\bar{\theta}$.
- Note:
  Money is neutral when a change in the quantity of money affects only the level of prices, and not the level of output or of other real variables.
  Money is superneutral – or long-run neutral – if changes in the steady-state rate of growth of the money supply do not affect the value of real economic variables (with the exception of money balances).

The Nonlinear Equilibrium

$$u_{C,t} = \beta R_t E_t u_{C,t+1} \tag{1}$$

$$u_{C,t} = \beta E_t \left(u_{C,t+1} \left(1 - \delta + f_{Kt} \right) \right) \tag{2}$$

$$0 = u_{L,t} + u_{C,t} f_{Lt} \tag{3}$$

$$u_{C,t} = u_{m,t} + \beta E_t \left(u_{C,t+1} \frac{1}{\Pi_{t+1}} \right) \tag{4}$$

$$M_t = \theta_t M_{t-1} \tag{5}$$

$$C_t + K_t = Y_t + (1 - \delta) K_{t-1} \tag{6}$$

$$Y_t = A_t f \left(K_{t-1}, L_t \right) \tag{7}$$

θ_t and A_t exogenous processes

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

$$\ln \theta_t = (1 - \rho) \ln \overline{\theta} + \rho \ln \theta_{t-1} + \varepsilon_t^M$$

Linearized

The log-linear will be given by:

$$\widehat{Y}_t = \alpha \widehat{K}_{t-1} + (1-\alpha)\,\widehat{L}_t + \widehat{A}_t \tag{L1}$$

$$\widehat{Y}_t = \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}\left(\widehat{K}_t - (1-\delta)\,\widehat{K}_{t-1}\right) \tag{L2}$$

$$\widehat{R}_t = \frac{\alpha\beta Y}{K}\left(E_t\widehat{Y}_{t+1} - \widehat{K}_t\right) \tag{L3}$$

$$\widehat{R}_t = \phi\left(E_t\widehat{C}_{t+1} - \widehat{C}_t\right) - b\,(1-\phi)\,(E_t\widehat{m}_{t+1} - \widehat{m}_t) \tag{L4}$$

$$\eta\widehat{L}_t = \widehat{Y}_t - \phi\widehat{C}_t + b\,(1-\phi)\,\widehat{m}_t \tag{L5}$$

$$\widehat{R}_t + E_t\widehat{\pi}_{t+1} = \frac{\overline{\theta}-\beta}{\beta}\left(\widehat{C}_t - \widehat{m}_t\right) \tag{L6}$$

$$\widehat{m}_t - \widehat{m}_{t-1} = \widehat{\theta}_t - \widehat{\pi}_t \tag{L7}$$

Separability: The Real Side

In the special case in which utility is separable in consumption and money balances, $\phi = 1$. This is an interesting case because now the equations of the model $(L1)$ to $(L7)$ can be separated in two independent blocks. The first block includes equations $(L1)$ to $(L5)$ adequately modified:

$$
\begin{aligned}
\widehat{Y}_t &= \alpha \widehat{K}_{t-1} + (1-\alpha)\widehat{L}_t + \widehat{A}_t \\
\widehat{Y}_t &= \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}\left(\widehat{K}_t - (1-\delta)\widehat{K}_{t-1}\right) \\
\widehat{R}_t &= \frac{\alpha\beta Y}{K} E_t\left(\widehat{Y}_{t+1} - \widehat{K}_t\right) \\
\widehat{R}_t &= \phi E_t\left(\widehat{C}_{t+1} - \widehat{C}_t\right) \\
\eta \widehat{L}_t &= \widehat{Y}_t - \phi \widehat{C}_t
\end{aligned}
$$

where can solve for C_t, K_t, Y_t, R_t and L_t independently of the rest of the model. Money is thus completely neutral for the real variables, in and out of the steady state.

Separability: The Nominal Side

Equations $(L6)$ and $(L7)$ study inflation and money growth independently of the real variables. To this end, assume technology is constant, so that the real interest rate and consumption will be constant too. Then, from $(L6)$:

$$\widehat{R}_t + E_t\widehat{\pi}_{t+1} = \frac{\overline{\theta} - \beta}{\beta}\left(\widehat{C}_t - \widehat{m}_t\right)$$

we obtain

$$\widehat{m}_t \equiv \widehat{M}_t - \widehat{P}_t = \frac{\beta}{\overline{\theta} - \beta}\left(\widehat{P}_t - E_t\widehat{P}_{t+1}\right)$$

this is Cagan money demand. If people expect high inflation in the future, they will reduce their real model holdings now. This equation can alternatively be solved for P_t

$$\widehat{P}_t = \frac{\beta}{\overline{\theta}}E_t\widehat{P}_{t+1} + \frac{\overline{\theta} - \beta}{\overline{\theta}}\widehat{M}_t \qquad (*)$$

Separability: Prices and Money

Above, we assume positive inflation in steady state (given by the steady state growth rate of M, so that $\Pi = \overline{\theta} \geq 1$). When we look at deviations from steady state, we perturb the path of money around the steady state in which nominal money is growing over time at rate $\overline{\theta}$, real money is constant, and there are innovations to θ_t at each point in time. Let $\sigma = \frac{\beta}{\overline{\theta}-\beta}$. The monetary side of our model can be written as:

$$\begin{aligned} \widehat{m}_t &= -\sigma E_t \widehat{\pi}_{t+1} && \text{(m1)} \\ \widehat{m}_t - \widehat{m}_{t-1} &= \widehat{\theta}_t - \widehat{\pi}_t && \text{(m2)} \end{aligned}$$

together with the linearized version of $\ln \theta_t = (1-\rho)\ln \overline{\theta} + \rho \ln \theta_{t-1} + \varepsilon_t$, namely

$$\widehat{\theta}_t = \rho \widehat{\theta}_{t-1} + \widehat{\varepsilon}_t$$

where $\widehat{\varepsilon}_t = \varepsilon_t$.

Separability: Prices and Money (continued)

Study a one-period, transitory increase in the growth rate of money with persistence ρ. From the expression above, we guess that a solution for inflation is:

$$\widehat{\pi}_t = \varepsilon_1 \widehat{m}_{t-1} + \varepsilon_2 \widehat{\theta}_t \tag{guess}$$

Using this guess, we get, using $E_t \widehat{\theta}_{t+1} = \rho \widehat{\theta}_t$ and plugging $(guess)$ into $(m1)$:

$$\widehat{m}_t = -\sigma E_t \left(\varepsilon_1 \widehat{m}_t + \varepsilon_2 \widehat{\theta}_{t+1} \right)$$
$$(1 + \sigma \varepsilon_1) \widehat{m}_t = -\sigma \varepsilon_2 \rho \widehat{\theta}_t$$
$$(1 + \sigma \varepsilon_1) \left(\widehat{m}_{t-1} + \widehat{\theta}_t - \widehat{\pi}_t \right) = -\sigma \varepsilon_2 \rho \widehat{\theta}_t$$
$$(1 + \sigma \varepsilon_1) \left(\widehat{m}_{t-1} + \widehat{\theta}_t - \varepsilon_1 \widehat{m}_{t-1} - \varepsilon_2 \widehat{\theta}_t \right) = -\sigma \varepsilon_2 \rho \widehat{\theta}_t$$

These two equations must hold for all values of $\widehat{m}_{t-1}$ and $\widehat{\theta}_t$, thus implying that the values of the undetermined coefficients ε_1 and ε_2 will be equal to:

$$\begin{aligned} 1 - \varepsilon_1 &= 0 \\ (1 + \sigma \varepsilon_1)(1 - \varepsilon_2) &= -\sigma \varepsilon_2 \rho \end{aligned}$$

Separability: Prices and Money (continued 2)

hence

$$\begin{aligned} \varepsilon_1 &= 1 \\ \varepsilon_2 &= \frac{1+\sigma}{1+\sigma(1-\rho)} \end{aligned}$$

Since σ is positive, the impact response of inflation to a shock in $\widehat{\theta}_t$ will be in general larger than one, so long as $\rho > 0$. This also tells us than in increase in $\widehat{\theta}_t$ leads to a fall in $\widehat{m}_t$.

$$\widehat{\pi}_t = \widehat{m}_{t-1} + \frac{1+\sigma}{1+\sigma(1-\rho)}\widehat{\theta}_t$$

Bottom line: how much inflation rises in response to a monetary expansion depends on how temporary or persistent the increase in money supply is, and on the underlying process for money. But, so long as prices are forward looking, an increase in the rate of money growth should cause an even larger response in inflation.

Real and Monetary Interactions

Additional things worth knowing

1. Real Effects of Monetary Shocks when ϕ is not 1
   (bottom line: when ϕ is not 1, money has some real effects, but many economists have that these effects are of ambiguous sign and extremely small)
2. Effect of Technology Shocks on Nominal Variables
   (what happens to nominal variables after real shocks depends on what the central bank does)
3. Taylor rule
   Given
   $$\widehat{R}_t + E_t \widehat{\pi}_{t+1} = \frac{\overline{\theta} - \beta}{\beta} \left(\widehat{C}_t - \widehat{m}_t \right)$$
   one can alternatively work with money supply or interest rate rules.

New-Keynesian Models

New-Keynesian Models layer nominal rigidities on top of the Money in Utility Model we saw before.
The main actors of the DNK model are:

agnt/mkt	final	interm.	K	L	profit	money	bonds	
HH	$-C$		$ZK_{-1} - qI$	$\frac{WL}{P}$	F	$\frac{M_{-1}-M+T}{P}$	$\frac{RB_{-1}-B}{P}$	$=0$
K F	$-I$		$q\phi(\cdot)K$					$=0$
final F	Y	$\frac{-\int_0^1 P_j Y_j dj}{P}$						$=0$
interm.F		$\frac{\int_0^1 P_j Y_j dj}{P}$	$-ZK_{-1}$	$\frac{-WL}{P}$	$\frac{1-X}{X}Y$			$=0$
G						$\frac{M-M_{-1}-T}{P}$		$=0$
equm	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	

where $I = K - (1-\delta) K_{-1}$

Actors

- households: make consumption and labor supply decisions, demand money and bonds. Buy installed capital at market price q and rent it out to intermediate firms which will use it as an intermediate input for production.
- capital producers: after production of Y, they make new capital goods after purchasing raw output. They sell the capital goods output to households.
- final good firms: produce final goods Y_t from intermediate goods Y_{jt}
- intermediate good firm: use labor and capital to produce intermediate goods Y_{jt}. Over each of this goods they have monopoly power. Can set price of good Y_j
- government: runs monetary policy.

Households

$$\max_{B_t, L_t, K_t, M_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\rho}}{1-\rho} - \frac{1}{\eta} (L_t)^{\eta} + \chi \ln \frac{M_t}{P_t} \right)$$

Let $\pi_t \equiv P_t / P_{t-1}$ denote the gross rate of inflation from period $t-1$ to period t. In real terms, the budget constraint is:

$$C_t + b_t + q_t K_t + \frac{M_t}{P_t} = \frac{R_{t-1}}{\pi_t} b_{t-1} + w_t L_t + F_t + \frac{M_{t-1}}{P_t} + T_t + Z_t K_{t-1} + q_t (1-\delta) K_{t-1}$$

where F_t denotes lump-sum dividends received from ownership of intermediate goods firms (whose problems are described below).

Household ctd

Solving this problem yields first order conditions for consumption/saving, labour supply, capital and money demand.

$$\frac{1}{C_t^{\rho}} = \beta E_t \left(\frac{R_t}{\pi_{t+1} C_{t+1}^{\rho}} \right) \qquad \text{(Euler)}$$

$$\frac{w_t}{C_t^{\rho}} = L_t^{\eta-1} \qquad \text{(LS)}$$

$$R_t = E_t \left(\pi_{t+1} \left(\frac{Z_{t+1} + q_{t+1} (1-\delta)}{q_t} \right) \right) \qquad \text{(KD)}$$

$$\frac{1}{C_t^{\rho}} = E_t \left(\beta \frac{1}{\pi_{t+1}} \frac{1}{C_{t+1}^{\rho}} \right) + \chi (m_t)^{-1} \qquad \text{(MD)}$$

where m_t is real balances (M_t/P_t), $w_t = W_t/P_t$, $b_t = B_t/P_t$ (remember, B is a nominal bond; if you divide it by P you get the nominal bond expressed in real terms, but you don't get a real bond).

Final Good Firms

Final good firm produces final good Y_t using intermediates Y_{jt}.
Total final goods are CES aggregator of different quantities of goods produced ($\varepsilon > 1$):

$$Y_t \leq \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Firm minimizes expenditure given production constraint. Lagrangean is:

$$L = \int_0^1 P_{jt} Y_{jt} dj + P_t \left(Y_t - \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)$$

Optimal choice of Y_{jt} solves for each j:

$$P_{jt} = P_t \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{1}{\varepsilon-1}} Y_{jt}^{-\frac{1}{\varepsilon}} \text{ and } Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t$$

This expression can be solved for the multiplier. Use the definition of Y_t with equality and use the solution for Y_{jt} to write:

$$Y_t = \left(\int_0^1 \left(\left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t \right)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} = Y_t \left(\int_0^1 \left(\frac{P_{jt}}{P_t} \right)^{1-\varepsilon} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Because the production function has constant returns to scale, Y_t drops from both sides of the expression, and we can then solve for P_t as:

$$P_t = \left(\int_0^1 P_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}.$$

P_t represents the minimum cost of achieving one unit of the final-goods bundle Y_t. For this reason we interpret P_t (the Lagrange multiplier) as the aggregate price index.

Intermediate Goods: Constraints

The intermediate goods sector is made by a continuum of monopolistically competitive firms owned by consumers, indexed by $j \in (0, 1)$. Each firm faces a downward sloping demand for its product. It produces output according to:[1]

$$Y_{jt} = A_t L_{jt}^{\alpha} K_{jt}^{1-\alpha}$$

Each producer chooses own sale price P_{jt} taking as given the demand curve. He can reset his price only when given the chance of doing so, which occurs with probability $1 - \theta$ in every period.
Intermediate goods firms face three constraints:

1. the production constraint;
2. the demand curve $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} Y_t$
3. prices can be adjusted only with probability $1 - \theta$

We can break this problem down into two sub-problems. As a cost minimizer and as a price setter.

[1] Capital is not predetermined at the firm level, hence we denote it with K_{jt}. At the aggregate level, it is predetermined, hence $\int K_{jt} dj = K_{t-1}$

Intermediate Goods: Cost Minimization

Consider cost minimization problem first, conditional on output Y_{jt}. This involves minimizing $\frac{W_t}{P_t}L_{jt} + Z_t K_{jt}$ subject to producing Y_{jt}. In real terms:

$$\min_{K_{jt},L_{jt}} \frac{W_t}{P_t}L_{jt} + Z_t K_{jt} + \mu_t \left(Y_{jt} - A_t L_{jt}^{\alpha} K_{jt}^{1-\alpha} \right) \qquad [\mu_t]$$

where μ_t is multiplier associated with the constraint (think μ_t as real marginal cost; define its inverse $X_t = 1/\mu_t$ as "markup"). The FOCs imply:

$$\alpha \frac{Y_{jt}}{L_{jt}} = \frac{1}{\mu_t}\frac{W_t}{P_t} \equiv X_t \frac{W_t}{P_t} \qquad \text{(LD)}$$

$$(1-\alpha)\frac{Y_{jt}}{K_{jt}} = \frac{1}{\mu_t} Z_t \equiv X_t Z_t \qquad \text{(KD)}$$

FOCs imply than we can write the real cost function as:

$$COST_{jt} = \underbrace{\frac{W_t}{P_t}L_{jt}}_{\alpha\mu_t Y_{jt}} + \underbrace{Z_t K_{jt}}_{(1-\alpha)\mu_t Y_{jt}} = \mu_t Y_{jt}$$

hence μ_t measures the real marginal cost for each producer.

Intermediate Goods: Price Setting 1

Average price level is CES aggregate of all prices:

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta)\left(P_t^*\right)^{1-\varepsilon} \tag{*}$$

P_{t-1} previous price level, P^* average price level chosen by those who can reset. Consider producer who can reset prices at time t. His demand curve is:

$$Y_{jt+k}^* = \left(P_{jt}^*/P_{t+k}\right)^{-\varepsilon} Y_{t+k}$$

for any k for which he will keep that price. Maximization problem is:

$$\max_{P_{jt}^*} \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k}\left(\frac{P_{jt}^*}{P_{t+k}} - \mu_{t+k}\right) Y_{jt+k}^*\right], \Lambda_{t,k} = \left(C_t/C_{t+k}\right)^{\rho}$$

where μ_t is the real marginal cost. θ is probability that P^* chosen at t will still apply later. The argument of the maximization problem is the "expected discounted sum of all profits that price setter will make conditional on chosen P_{jt}^* and weighted by how likely P_{jt}^* is to stay in place in future periods".

Intermediate Goods: Price Setting 2

At t, price setter chooses P^*_{jt} to maximize profit. Differentiate wrt P^*_{jt} to get:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \left(\frac{Y^*_{jt+k}}{P_{t+k}} + \frac{P^*_{jt}}{P_{t+k}} \frac{\partial Y^*_{jt+k}}{\partial P^*_{jt}} - \mu_{t+k} \frac{\partial Y^*_{jt+k}}{\partial P^*_{jt}} \right) \right] = 0$$

take Y^*_{jt+k}/P_{t+k} out, isolate elasticity of Y^*_{jt+k} wrt P^*_{jt}

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y^*_{jt+k}}{P_{t+k}} \left(1 + \frac{P^*_{jt}}{Y^*_{jt+k}} \frac{\partial Y^*_{jt+k}}{\partial P^*_{jt}} - \frac{\mu_{t+k} P_{t+k}}{P^*_{jt}} \frac{P^*_{jt}}{Y^*_{jt+k}} \frac{\partial Y^*_{jt+k}}{\partial P^*_{jt}} \right) \right] = 0$$

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y^*_{jt+k}}{P_{t+k}} \left(1 - \varepsilon + \frac{\mu_{t+k} P_{t+k}}{P^*_{jt}} \varepsilon \right) \right] = 0$$

multiply everything by P^*_{jt}

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y^*_{jt+k}}{P_{t+k}} \left(P^*_{jt} - \frac{\varepsilon}{\varepsilon - 1} \mu_{t+k} P_{t+k} \right) \right] = 0$$

In the symmetric equilibrium, all the firms that reset the price choose the same price (and face the same demand), hence

$$P_{jt}^{*} = P_{t}^{*}$$

These two expressions enter the equilibrium (using $X = \frac{\varepsilon}{\varepsilon - 1}$ = steady state markup). One is $*$ that we derived above, the other is:

$$\sum_{k=0}^{\infty} (\theta\beta)^{k} E_{t}\left[\Lambda_{t,k} Y_{t+k}^{*}\left(\frac{P_{t}^{*}}{P_{t+k}} - X\mu_{t+k}\right)\right] = 0 \qquad \text{(**)}$$

Intermediate Goods: Price Setting 3

Use $\mu^n_{t+k} \equiv \mu_{t+k} P_{t+k}$ to rearrange the expression above to obtain:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} \frac{P^*_t}{P_{t+k}} \right] = X \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} \mu_{t+k} \right]$$

$$P^*_t = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} \mu^n_{t+k} P^{-1}_{t+k} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k} \right]} = X \sum_{k=0}^{\infty} \phi_{t,k} \mu^n_{t+k}$$

where $\phi_{t,k} \equiv \frac{(\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k} \right]}$. This expression says that the optimal price is a weighted average of current and expected future nominal marginal costs. Weigths depend on expected demand in the future, and how quickly firm discounts profits.

Therefore you can notice the following:

- Under purely flexible prices, $\theta = 0$: the markup is a constant. $P^*_t = X\mu^n_t$ and optimal prices are a multiple X of the marginal cost.
- When $\theta > 0$, the optimal price depends on future expected values of aggregate variables as well as future nominal marginal costs μ^n_{t+k}.

Phillips Curve: Price Setting 4

Use $Y_{t+k}^{*} = \left(P_t^{*}/P_{t+k}\right)^{-\varepsilon} Y_{t+k}$ and cancel out P_t^{*} in num/den to obtain:

$$P_t^{*} = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} P_{t+k}^{\varepsilon-1} Y_{t+k} \mu_{t+k}^{n} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} P_{t+k}^{\varepsilon-1} Y_{t+k} \right]}$$

Loglinearize now. Numerator and denominator differ up to multiple μ_{t+k}^{n}, which multiplies $(\theta\beta)^k (1-\theta\beta)$. Hence expect that in log-linearising Y, $P^{\varepsilon-1}$, Λ will cancel out and disappear. Rearranging and dividing by P_t:

$$\frac{P_t^{*}}{P_t} \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k} P_{t+k}^{\varepsilon-1} \right] = \frac{1}{P_t} X \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k} P_{t+k}^{\varepsilon-1} \mu_{t+k}^{n} \right]$$

LHS: $\left(\widehat{P}_t^{*} - \widehat{P}_t \right) \sum (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1} \right] + \sum (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1} \right] E_t \left(\widehat{\Lambda}_{t,k} + \widehat{Y}_{t+k} + (\varepsilon-1) \widehat{P}_{+k} \right)$

RHS: $-\widehat{P}_t \sum (\theta\beta)^k \Lambda Y P^{\varepsilon-1} + \frac{X}{P} \sum (\theta\beta)^k \left[\frac{\Lambda Y P^{\varepsilon-1} P}{X} \right] E_t \left(\widehat{\Lambda}_{t,k} + \widehat{\mu^n}_{t+k} + \widehat{Y}_{t+k} + (\varepsilon-1) \widehat{P}_{+k} \right)$

$= -\widehat{P}_t \sum (\theta\beta)^k \Lambda Y P^{\varepsilon-1} + \sum (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1} \right] E_t \left(\widehat{\Lambda}_{t,k} + \widehat{\mu^n}_{t+k} + \widehat{Y}_{t+k} + (\varepsilon-1) \widehat{P}_{+k} \right)$

(use $\mu^n = P/X$ in steady state, and $\sum$ runs from 0 to ∞)

Hence, using $\widehat{\mu^n}_{t+k} = \widehat{P}_{t+k} + \widehat{\mu}_{t+k}$

$$\widehat{P}_t^* \sum_{k=0}^{\infty} (\theta\beta)^k = \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left(\widehat{P}_{t+k} + \widehat{\mu}_{t+k} \right)$$

$$\widehat{P}_t^* = (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left(\widehat{P}_{t+k} + \widehat{\mu}_{t+k} \right) \qquad (@)$$

@ states that optimal price equals average of current and future marginal costs, weighted by probability that price will hold in later periods. So you assign weight 1 to today, $\theta\beta$ to tomorrow, $\theta^2\beta^2$ to after tomorrow, and so on. Notice the forwardlookingness, and that weights are normalized so to sum up to one.

But we know that:

$$\begin{aligned}
\widehat{P}_t - \theta \widehat{P}_{t-1} &= (1-\theta)\,\widehat{P}_t^* \\
\widehat{P}_t - \theta \widehat{P}_{t-1} &= (1-\theta)(1-\theta\beta)\,E_t\left(\left(\widehat{P}_t + \widehat{\mu}_t\right) + \theta\beta\left(\widehat{P}_{t+1} + \widehat{\mu}_{t+1}\right) + \theta^2\beta^2(...) + ...\right) \\
\widehat{P}_t - \theta \widehat{P}_{t-1} &= (1-\theta)(1-\theta\beta)\left(\widehat{P}_t + \widehat{\mu}_t\right) + \theta\beta\left(E_t\widehat{P}_{t+1} - \theta\widehat{P}_t\right) \\
&\text{...and after some algebra....} \\
\widehat{\pi}_t &= (1-\theta)\,\widehat{\pi}_t + \theta\beta E_t\widehat{\pi}_{t+1} + (1-\theta)(1-\theta\beta)\,\widehat{\mu}_t
\end{aligned}$$

$$\widehat{\pi}_t = \beta E_t \widehat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{\mu}_t$$

This equation Phillips curve: inflation rises when the real marginal costs rise. Note that:

- the higher β, the higher the weight to future $\widehat{\mu}_t$'s, and the lower today's elasticity to current marginal cost
- the higher θ, the higher the chance that I will be stuck with my price for a long period, and the higher the elasticity of $\widehat{P}_t^*$ to $\widehat{\mu}_t$. However, few prices will be changed in the aggregate, aggregate inflation will not be sensitive to the marginal cost.

Capital Producers

Competitive capital producers purchase raw output as materials input I_{jt}, *rent capital within period* from intermediate firms and produce new capital goods sold at price q_t. The production function for new capital is given by:

$$Y_{jt}^{k} = \Phi\left(I_{jt}/K_{jt}\right) K_{jt}$$

$$\Phi' > 0,\ \ \Phi'' < 0,\ \ \Phi(0) = 0,\ \ \Phi\left(\frac{I}{K}\right) = \frac{I}{K}$$

These properties imply:
1) constant returns to scale in I and K
2) diminishing returns to I, holding K constant
The representative firm solves:

$$\max_{I_{jt},K_{jt}} q_t \Phi(\cdot) K_{jt} - I_{jt} - Z_t^k K_{jt}$$

it purchases raw output at 1, rents installed capital at Z^k and produces new capital valued at q using the technology $\Phi(\cdot) K$.

Capital Producers (continued)

The first order conditions for I_{jt} and K_{jt} are:

$$
\begin{aligned}
q_t &= 1/\Phi'\left(\frac{I_{jt}}{K_{jt}}\right) \\
q_t\left(\Phi - \Phi'\frac{I_{jt}}{K_{jt}}\right) &= Z_t^k
\end{aligned}
$$

The first condition implies that as I/K rises, Φ' falls and q rises.
The condition for K guarantees that in steady state, since $\Phi\left(I/K\right) = I/K$, $\Phi'\left(I/K\right) = 1$, which implies that $Z^k = 0$ and can be ignored.

Aggregation

Total output in economy is:

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} = \left[\int_0^1 \left(A_t K_{jt}^{1-\alpha} L_{jt}^{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

It is not possible to simplify this expression since input usages across firms differs. However the linear aggregator:

$$Y_t' = \int_0^1 Y_{jt} dj$$

is approximately equal to Y_t within a local region of the steady state. Hence for local analysis we can simply use:

$$Y_t = A_t K_{t-1}^{1-\alpha} L_t^{\alpha},\ L_t = \int_0^1 L_{jt} dj,\ \ K_{t-1} = \int_0^1 K_{jt} dj = C_t + I_t$$

Capital goods mkt clearing

$$K_t = \Phi\left(I_t / K_{t-1}\right) K_t + (1-\delta) K_{t-1}$$

Bond market clearing implies $B_t = 0$. CB policy sets the dynamics of money supply in some fashion and closes the model.

Equilibrium in Levels

Equation summarizing the model (see file `dnknotes.mod`)

$$Y_t = C_t + I_t \quad (1)$$

$$\frac{1}{C_t^{\rho}} = \beta \left(\frac{R_t P_t}{C_{t+1}^{\rho} P_{t+1}} \right) \quad (2)$$

$$q_t = 1/\Phi' \left(\frac{I_t}{K_{t-1}} \right) \quad (3)$$

$$E_t \left(\frac{R_t}{\pi_{t+1}} \right) = \frac{1}{q_t} E_t \left(\frac{(1-\alpha) Y_{t+1} \mu_{t+1}}{K_t} + q_{t+1} (1-\delta) \right) \quad (4)$$

$$Y_t = A_t L_t^{\alpha} K_{t-1}^{1-\alpha} \quad (5)$$

$$\alpha \frac{Y_t}{L_t} = \frac{1}{\mu_t} L_t^{\eta-1} C_t^{\rho} \quad (6)$$

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta) \left(X \sum_{k=0}^{\infty} \phi_{t,k} \mu_{t+k} P_{t+k} \right)^{1-\varepsilon} \quad (7)$$

$$K_t = \Phi \left(I_t / K_{t-1} \right) K_t + (1-\delta) K_{t-1} \quad (8)$$

$$R_t = R_{t-1}^{\phi_r} rr^{1-\phi_r} \left(\left(P_t / P_{t-1} \right)^{1+\phi_\pi} \mu_t^{\phi_z} \right)^{1-\phi_r} \varepsilon_{r,t} \quad (9)$$

Equilibrium in Levels 2

1. market clearing
2. aggregate demand equation
3. supply of capital
4. equilibrium rental rate of K
5. production function
6. equilibrium in the labor market. Take labour demand (LD) and labor supply (LS) and impose market clearing. Then equate (LD) and (LS) so as to eliminate of the real wage w from that expression.
7. aggregate price level is a weighted average of (1) previous price level P_{t-1} and (2) reset prices P_t^*, which depend on future expected marginal costs.
8. capital dynamics
9. monetary policy rule. CB sets nominal interest rate to be a function of previous interest rate, current inflation and current marginal costs. This is a Taylor rule. $\varepsilon_{r,t}$ is monetary policy shock. rr is target level of the interest rate: assuming zero inflation in steady state is tantamount to assume that steady state nominal interest rate equals real interest rate rr.

Log-Linearized Model

We linearize around a steady state with zero inflation and make use of the following

$$\begin{aligned} q &= 1 \\ R &= \frac{1}{\beta} = (1-\alpha)\frac{Y}{K}\mu + 1 - \delta \\ \frac{C}{Y} &= 1 - \delta\frac{K}{Y} \end{aligned}$$

Also denote (for later)

$$\begin{aligned} \varphi &= -\frac{\Phi'' \frac{I}{K}}{\Phi'} \\ \zeta &= \frac{(1-\theta)(1-\beta\theta)}{\theta} \end{aligned}$$

LL Model 2

$$\widehat{Y}_t = \frac{C}{Y}\widehat{C}_t + \frac{I}{Y}\widehat{I}_t$$

$$-\rho\left(\widehat{C}_t - E_t\widehat{C}_{t+1}\right) = \widehat{R}_t - \widehat{\pi}_{t+1}$$

$$\widehat{q}_t = -\widehat{R}_t + E_t\widehat{\pi}_{t+1} + \beta\left(1-\delta\right)E_t\widehat{q}_{t+1} + \left(1-\beta\left(1-\delta\right)\right)E_t\left(\widehat{\mu}_{t+1} + \widehat{Y}_{t+1} - \widehat{K}_t\right)$$

$$\widehat{q}_t = \varphi E_t\left(\widehat{I}_t - \widehat{K}_{t-1}\right)$$

$$\widehat{Y}_t = \widehat{a}_t + \alpha\widehat{L}_t + \left(1-\alpha\right)\widehat{K}_{t-1}$$

$$\widehat{Y}_t + \widehat{\mu}_t - \rho\widehat{C}_t = \eta\widehat{L}_t$$

$$\widehat{\pi}_t = \beta E_t\widehat{\pi}_{t+1} + \zeta\widehat{\mu}_t$$

$$\widehat{K}_t = \delta\widehat{I}_t + \left(1-\delta\right)\widehat{K}_{t-1}$$

$$\widehat{R}_t = \phi_R\widehat{R}_{t-1} + \left(1-\phi_R\right)\left(\left(1+\phi_\pi\right)\widehat{\pi}_t + \phi_\mu\widehat{\mu}_t\right) + \widehat{e}_t$$

Natural Variables

- Convenient define Y_t^* as the natural output under completely flexible prices $(\theta = 0)$.
- This way μ_t can in fact be eliminated. If firms can change prices optimally each period, $\mu_t = 0$ (since $\zeta \to \infty$).
- Without capital $(\alpha = 1)$, we can easily derive an expression for μ as a function of the gap between flexible price and sticky price equilibrium, that is:

$$\widehat{\mu}_t = (\eta + \rho - 1)\left(\widehat{Y}_t - \widehat{Y}_t^*\right) = \lambda\left(\widehat{Y}_t - \widehat{Y}_t^*\right)$$

  hence the real marginal cost is positive whenever Y is above Y^*.
- Here Y^* is an exogenous variable, since it depends only on technology, which is exogenous. In fact, setting $\mu = 0$ and solving for Y^* yields:

$$\widehat{Y}_t^* = \frac{\eta}{\eta + \rho - 1}\widehat{a}_t$$

The 3-equation NK model

With this convention, the dynamic-new keynesian model becomes:

$$\begin{aligned}
\widehat{Y}_t &= E_t\widehat{Y}_{t+1} - \rho^{-1}\left(\widehat{R}_t - E_t\widehat{\pi}_{t+1}\right) + \widehat{g}_t && \text{(a)}\\
\widehat{\pi}_t &= \lambda\zeta\left(\widehat{Y}_t - \widehat{Y}_t^*\right) + \beta E_t\widehat{\pi}_{t+1} + \widehat{u}_t && \text{(b)}\\
\widehat{R}_t &= \Phi_R\widehat{R}_{t-1} + (1-\Phi_R)\left((1+\Phi_\pi)\,\widehat{\pi}_t + \Phi_x\left(\widehat{Y}_t - \widehat{Y}_t^*\right)\right) + \widehat{e}_t && \text{(c)}
\end{aligned}$$

where g, u and e are “shocks”

Insights from DNK Model

- Technology and Employment
- Dynamics following a monetary shock
- Extensions
- Drivers of Inflation and the Slope of the Phillips Curve

Policy Rules and Indeterminacy

SImplify and modify model as follows:

$$
\begin{aligned}
R_t &= (1+\phi_\pi)\,\pi_t + \phi_x x_t + e_t \\
\pi_t &= \kappa x_t + \beta E_t \pi_{t+1} + u_t \\
x_t &= E_t x_{t+1} - \sigma(R_t - E_t \pi_{t+1}) + g_t
\end{aligned}
$$

3-equation NK model as in Clarida, Gali, and Gertler (QJE, 1997)

Once you eliminate R the system becomes (Woodford, Appendix C, page 677):

$$\begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} E\pi_{t+1} \\ Ex_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma\left(1+\phi_\pi\right) & 1+\phi_x\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + Mz_t$$

$$\begin{bmatrix} E\pi_{t+1} \\ Ex_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma\left(1+\phi_\pi-\beta^{-1}\right) & 1+\sigma\phi_x+\frac{\kappa\sigma}{\beta} \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix}^{-1} Mz_t$$

With a fair amount of matrix algebra you can show that the eigenvalues are both greater than one in modulus if:

$$\phi_\pi + \frac{1-\beta}{\kappa}\phi_x > 0$$

This condition can be easily interpreted. Observe that from the Phillips curve a 1% rise in inflation implies that the output gap rises by $\frac{1-\beta}{\kappa}$%. Hence for determinacy the nominal interest rate must rise by at least $\phi_\pi - 1$ to offset the rise in inflation, and by $\frac{1-\beta}{\kappa}$ to offset the rise in the output gap. Otherwise each increase in inflation would be self-fulfilling.

Remark

In a model of this kind an interest rate peg results in indeterminacy of the equilibrium. In other words, there are an infinite number of possible responses of the variables to real disturbances, including some in which fluctuations in output and inflation are disproportionately large relative to the size of the disturbances. Hence this implies that the central bank must respond to the real variables to guarantee determinacy of equilibrium.

ZLB

ZLB implies the following:

$$\begin{aligned} R_t &= \max\left(-rzlb, (1+\phi_\pi)\,\pi_t + \phi_x x_t + e_t\right) \\ \pi_t &= \kappa x_t + \beta E_t \pi_{t+1} + u_t \\ x_t &= E_t x_{t+1} - \sigma(R_t - E_t \pi_{t+1}) + g_t \end{aligned}$$

where *rzlb* is the steady state nominal interest rate.
See `Occbin_dnk3eq`

A 2-country Open Economy Model

How Do We Go from One to Two countries?

1. Specify Goods Produced
2. Specify Nature of Assets and Goods Traded Across Countries
3. Specify Monetary Policies

Key References

- Walsh, Chapter 9
- Gali, in turn based on Gali–Monacelli for small-open economy
- Lubik-Schorfheide (NBER) for a classic two country model http://www.nber.org/chapters/c0071.pdf
- Corsetti-Dedola-Leduc (Handbook of Monetary Economics) https://sites.google.com/site/giancarlocorsetti/main/handbook-codele.pdf?attredirects=0

Households

- Choose consumption and hours

$$\max \sum_{t=0}^{\infty} E_t \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - N_t \right)$$

$$C_t = \left((1-\alpha)^{\frac{1}{\eta}} \left(C_{H,t}\right)^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} \left(C_{F,t}\right)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}$$

η EOS,α import share

- Households allocate expenditures according to

$$\begin{aligned} C_{H,t} &= (1-\alpha)\left(P_{H,t}/P_t\right)^{-\eta} C_t \\ C_{F,t} &= \alpha \left(P_{F,t}/P_t\right)^{-\eta} C_t \\ P_t &= \left((1-\alpha) P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta} \right)^{\frac{1}{1-\eta}} \quad \text{(CPI)} \end{aligned}$$

- Their constraint is

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} + B_t + e_t B_t^* = W_t N_t + D_t - T_t + R_{t-1}B_{t-1} + R_{t-1}^* e_t B_{t-1}^*$$

we assume incomplete international asset markets

Optimality Conditions

- Let

$$L = E_t\Big(\frac{C_t^{1-\sigma}}{1-\sigma} - N_t + \beta\frac{C_{t+1}^{1-\sigma}}{1-\sigma} - \beta N_{t+1} + ...$$
$$-\lambda_t\left(P_{H,t}C_{H,t} + P_{F,t}C_{F,t} + B_t + e_tB_t^* - W_tN_t - R_{t-1}B_{t-1} - e_tR_{t-1}^*B_{t-1}^*...\right)/P_t$$
$$-\beta\lambda_{t+1}\left(... - R_tB_t - e_{t+1}R_t^*B_t + ...\right)\Big)/P_{t+1}$$

- The optimality are $(C_H, C_F, N_t, B_t, B_t^*)$

$$C_t^{-\sigma}\frac{dC_t}{dC_{H,t}} = \lambda_tP_{H,t} \quad (1)$$

$$C_t^{-\sigma}\frac{dC_t}{dC_{F,t}} = \lambda_tP_{F,t} \quad (2)$$

$$1 = \lambda_tW_t \quad (3)$$

$$\lambda_t = \beta E_t\left(\lambda_{t+1}\frac{R_t}{\pi_{t+1}}\right) \quad (4)$$

$$e_t\lambda_t = \beta E_t\left(\lambda_{t+1}e_{t+1}\frac{R_t^*}{\pi_{t+1}}\right) \quad (5)$$

Implications

- Combining 4 and 5, we get, up to a first order

$$r_t - r_t^* = E_t \Delta e_{t+1}$$

  Uncovered interest parity: equation linking interest rate differentials across economies under the assumption of financial market integration.

- Alternatively, you could regress Δe_t on a constant and $\left(r_{t-1} - r_{t-1}^*\right)$, and under rational expectations you should coefficient of $(a = 0, b = 1)$ on the two regressors

- Typically, you get negative b's from the data
  You can justify negative b assuming that the r's are correlated with current e_t. For instance, if

$$r_t - r_t^* = \mu e_t$$

  then a regression of $e_t - e_{t-1}$ will recover a coefficient of $-1/\mu$ on $r_t - r_t^*$ (see Walsh, Chapter 6)

Producers

- Domestically produced goods are a CES aggregate of varieties

$$Y_{H,t} = \int_0^1 \left(Y_{H,t}\left(j\right)^{1-\omega}\right)^{\frac{1}{1-\omega}} dj$$

thus resulting in

$$Y_{H,t} = \left(\frac{P_{H,t}\left(j\right)}{P_{H,t}}\right)^{-\omega} \left(C_{H,t} + C_{H,t}^{*}\right)$$

- Domestic firms produced according to

$$Y_{H,t}\left(j\right) = A_t N_t\left(j\right)$$

These firms face Calvo-style rigidity

Domestic Importers

- Domestic consumers can only buy foreign goods from importers who have market power.
- Pass-through from exchange rates to domestic currency prices of imports is imperfect because importers adjust pricing behavior to extract maximum revenue from consumers
- Importers cost of goods $e_t P^*_{F,t}(j)$. Importers revenue from selling them: $P_{F,t}(j)$
- Importers demand curve $C_{F,t}(j) = (P_{F,t}(j) / P_{F,t})^{-\omega} C_{F,t}$
- This implies that PPP does not hold. $P_{F,t}(j) \neq e_t P_{F,t}(j)$

Foreign Economy

The mirror image of the domestic economy

Some Important Definitions

1. The nominal exchange rate in absence of imperfect pass-through

$$\widetilde{e_t} = \frac{P_{F,t}}{P^*_{F,t}}$$

where $P_{F,t}$ = price_of_foreign_good_produced_in_US
$P^*_{F,t}$ = price of foreign good produced abroad

2. The real exchange rate

$$s_t = e_t \frac{P^*_t}{P_t}$$

with $s^*_t = 1/s_t$

3. The terms of trade

$$q_t = \frac{P_{H,t}}{P_{F,t}}$$

where $P_{H,t}$ are domestically produced goods, $P_{F,t}$ are foreign goods

4. The law-of-one-price gap

$$\psi_t = \frac{e_t P^*_{F,t}}{P_{F,t}}$$

Equilibrium

- Market clearing implies

$$\begin{aligned} Y_{H,t} &= C_{H,t} + C^*_{H,t} \\ Y^*_{F,t} &= C_{F,t} + C^*_{F,t} \end{aligned}$$

- Is consumption equalized?

$$\begin{aligned} \lambda_t &= \beta E_t \left(\lambda_{t+1} R_t / \pi_{t+1} \right) && \text{(a)} \\ e_t \lambda_t &= \beta E_t \left(\lambda_{t+1} e_{t+1} R^*_t / \pi_{t+1} \right) && \text{(b)} \\ \lambda^*_t &= \beta E_t \left(\lambda^*_{t+1} R^*_t / \pi^*_{t+1} \right) && \text{(c)} \\ \frac{\lambda^*_t}{e_t} &= \beta E_t \left(\frac{\lambda^*_{t+1}}{e_{t+1}} R_t / \pi^*_{t+1} \right) && \text{(d)} \end{aligned}$$

combine a and d to get (similar for b+c)

$$E_t \left(\frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\pi_{t+1}} \right) = \beta E_t \left(\frac{\lambda^*_{t+1}}{\lambda^*_t} \frac{e_t}{e_{t+1}} \frac{R_t}{\pi^*_{t+1}} \right)$$

Asset market structure implies expected consumption growth is equalized across countries (modified risk sharing condition; see Corsetti-Dedola-Leduc, Handbook, equation 26).

Log-linearizations, Domestic Block (see 3.6 in Lubik-Schorfheide)

$$\pi_{H,t} = \beta\pi_{H,t+1} + \kappa_H mc_{H,t} \quad \text{(price-domestic)}$$

$$mc_{H,t} = -\lambda_t - \alpha q_t - A_t \quad \text{(marg.cost)}$$

$$-\lambda_t = \sigma C_t \quad \text{(mu of wealth)}$$

$$\lambda_t = \lambda_{t+1} - R_t + \pi_{t+1} \quad \text{(consumption\_euler)}$$

$$\pi_{F,t} = \beta\pi_{F,t+1} + \kappa_F \psi_{F,t} \quad \text{(price-importers)}$$

$$\pi_t = \alpha\pi_{H,t} + (1-\alpha)\,\pi_{F,t} \quad \text{(CPI definition)}$$

$$q_t = q_{t-1} + \pi_{H,t} - \pi_{F,t} \quad \text{(terms of trade)}$$

$$s_t = \psi_{F,t} - \left((1-\alpha)\,q_t + \alpha q_t^*\right) \quad \text{(rer)}$$

$$\Delta e_t = \pi_t - \pi_t^* - \Delta s_t \quad \text{(nominal\_xr)}$$

$$R_t - R_t^* = E_t\Delta e_{t+1} \quad \text{(UIP)}$$

$$E_t\Delta\lambda_{t+1} - E_t\Delta\lambda_{t+1}^* = E_t\pi_{t+1}^* - E_t\pi_{t+1} - E_t\Delta e_{t+1} \quad \text{(imperfect risk sharing)}$$

$$y_{H,t} = c_t - \alpha s_t/\sigma - \alpha\,(1-\alpha)\,\eta\,(q_t - q_t^*) \quad \text{(market clearing)}$$

$$R_t = \phi_1\pi_t + \phi_2\Delta y_t + \phi_3\Delta e_t \quad \text{(policy rule)}$$

Foreign block follows same structure.

A simple 3-period economy with frictions

- $t = 1, 2, 3$. Two goods: consumption c and housing h. Housing in fixed supply. Agents produce of consumption goods using:

$$y_t = A_t F\left(h_{t-1}\right) = A_t h_{t-1}^{\gamma}$$

A_t stochastic in period 2 only (can be low or high)

- Two agents: utility functions are given by:

$$U^b\left(c_t\right) = E_1 \sum_{t=1}^{3} \beta^t \log\left(c_t\right), U^s\left(c_t'\right) = E_1 \sum_{t=1}^{3} \left(\beta'\right)^t \log\left(c_t'\right)$$

- Budget constraint for an agent that enters time t with h_{t-1} housing goods and b_{t-1} loans is given by, in each of three periods:

$$c_t + q_t h_t + R_{t-1} b_{t-1} \leq \omega_t + A_t h_{t-1}^{\gamma} + q_t h_{t-1} + b_t$$

  Where ω_t is a deterministic endowment.

- Collateral constraint is

$$b_t \leq m q_t h_t$$

The Lender's Problem

Given an initial endowment h'_{-1} and indebtedness $b'_{-1}R'_{-1}$, lender chooses $\{c'_t\}$ $\{b'_t\}$ and $\{h'_t\}$ to solve

$$\max E_1 \sum_{t=1}^{3} \beta'^t \log\left(c'_t\right)$$

subject to:

$$c'_t + q_t h'_t + R_{t-1} b'_{t-1} \leq \omega'_t + A'_t h'^{\gamma}_{t-1} + q_t h'_{t-1} + b'_t \tag{1}$$

The optimlaity conditions are given by

$$1 = \beta E_t \left\{ \frac{c'_t}{c'_{t+1}} \right\} R_t \tag{2}$$

$$1 = \beta E_t \left\{ \frac{c'_t}{c'_{t+1}} \frac{A_{t+1} F'\left(h'_t\right) + q_{t+1}}{q_t} \right\} \tag{3}$$

plus budget constraint at equality. The optimality conditions indicate that since lender is unconstrained he will equalize discounted return on housing and loans.

The Borrower's Problem

Given h_{-1} and $b_{-1}R_{-1}$, borrower chooses c_t, b_t and h_t to:

$$\max E_1 \sum_{t=1}^{3} \beta^t \log\left(c_t\right)$$

subject to:

$$c_t + q_t h_t - b_t \leq \omega_t + A_t h_{t-1}^{\gamma} + q_t h_{t-1} - b_{t-1} R_{t-1} \tag{4}$$

$$b_t \leq m q_t h_t \tag{5}$$

Letting λ_t denote lagrange multiplier on collateral constraint (5), we have:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{R_t}{c_{t+1}} \right\} + \lambda_t \tag{6}$$

$$\frac{q_t}{c_t} = \beta E_t \left\{ \frac{A_{t+1} F'\left(h_t\right) + q_{t+1}}{c_{t+1}} \right\} + m \lambda_t q_t \tag{7}$$

together with complementary slackness condition on (5) and budget constraint (4) at equality.

Implications of Binding Collateral Constraint

- Binding collateral constraint, $\lambda_t > 0$, prevents borrower from undertaking investment even if marginal benefit of such investment is greater then marginal cost of obtaining funds to finance it:

$$\beta E_t \left\{ \frac{c_t}{c_{t+1}} \frac{A_{t+1} F'(h_t) + q_{t+1}}{q_t} \right\} > \beta E_t \left\{ \frac{c_t}{c_{t+1}} R_t \right\}$$

- The collateral constraint is therefore preventing mutually beneficial trade between borrower's and savers.
- Welfare analysis below we will explore different ways in which a planner, although forced to respect collateral constraint and to operate through same markets as private agents, can reduce extent of such unexploited trade opportunities.

Equilibrium

- An equilibrium is an allocation $\{\{c_t\},\{c_t'\},\{h_t\},\{h_t'\},\{b_t\},\{b_t'\}\}$ and prices $\{\{R_t\},\{q_t\}\}$ such that given prices and initial conditions allocation solves agents' problem and all markets clear. The equilibrium allocation and prices are determined by system of equations $(1)-(7)$ plus complementary slackness condition on collateral constraint (5)
- There are two sources of inefficiency in this model: market incompleteness and collateral constraint. Our welfare analysis will take as a benchmark allocation chosen by a Pareto Planner that cannot undo either of these inefficiencies. This allows us to isolate effect of uninternalized pecuniary externalities on equilibrium allocation and welfare.
- Consider optimal regulation of borrowers' behavior.
  Assume planner can only prescribe allocation to borrower while not able to control behavior of lender. That is, planner is constrained to respect optimality conditions of lenders. In fact, these optimality conditions will represent supply schedules for loans, housing goods and consumption goods that planner uses in order to compute market prices associated to each allocation prescribed to borrower.

Planner's Problem

The planner problem is to choose an allocation $\{\{c_t\}, \{c'_t\}, \{h_t\}, \{h'_t\}, \{b_t\}\}$ and prices $\{\{R_t\}, \{q_t\}\}$ to solve

$$\max E_1 \sum_{t=1}^{3} \beta^t \log(c_t) \tag{8}$$

subject to:

$$c_t + q_t h_t - b_t \le \omega_t + A_t h_{t-1}^{\gamma} + q_t h_{t-1} - b_{t-1} R_{t-1} \tag{9}$$

$$b_t \le m q_t h_t \tag{10}$$

$$c'_t + q_t h'_t + b'_t \le \omega'_t + A_t h'^{\gamma}_{t-1} + q_t h'_{t-1} - b'_{t-1} R_{t-1} \tag{11}$$

$$1 = \beta E_t \left\{ c'_t / c'_{t+1} \right\} R_t \tag{12}$$

$$1 = \beta E_t \left\{ \frac{c'_t}{c'_{t+1}} \frac{A_{t+1} F'(h'_t) + q_{t+1}}{q_t} \right\} \tag{13}$$

$$E_1 \sum_{t=1}^{3} \beta'^t \log(c'_t) \ge v'^{CE}(h_{-1}, h'_{-1}, b_{-1} R_{-1}) \tag{14}$$

where $v'^{CE}(h_{-1}, h'_{-1}, b_{-1} R_{-1})$ is indirect utility function of lender in a competitive equilibrium with an initial distribution of wealth given by $(h_{-1}, h'_{-1}, b_{-1} R_{-1})$. The last constraint hence ensures that lender gets at least as much utility as it would in a *laissez-faire* equilibrium.

Definition of Constrained Inefficiency

- The competitive equilibrium is constrained inefficient if value obtained by borrower in Planner's allocation, optimized objective (5) , is higher than his indirect utility function in a competitive equilibrium with same initial distribution of wealth, $v^{CE}\left(h_{-1}, h'_{-1}, b_{-1}R_{-1}\right)$.
- In constrained efficient planners problem, if all prices (q, R) can be set arbitrarily, then planner achieves first-best solution. This result holds regardless of whether borrowing constraint is present or not.
- In constrained efficient planners problem, if planner does not internalize effects on prices, and if there are no externalities other than those on prices, then solution coincides with competitive equilibrium. This result holds as long as minimum amount of utility given to saver is consistent with competitive equilibrium.

Results. Is the Equilibrium Inefficient?

- Result 1: No uncertainty –> underborrowing, unless m is large or initial allocations are equal.
  underborrowing –> optimal policy wants to relax borrowing constraint. This can occur via subsidy on borrowing financed by lump-sum taxation in same period.
- Result 2: Uncertainty –> overborrowing or underborrowing.
  Overborrowing reflects **pecuniary externality**. Agents fail to internalize effect of decisions on prices –> inefficient allocation of resources.
  Intermediate values of endowment imply that borrowing constraint is slack in $t = 1$, but may bind in $t = 2$ and A_{LOW}.
  With **overborrowing**, optimal policy is **macroprudential**: it taxes borrowing on average - to avoid overborrowing if constraints are expected to bind. Then, if crisis hits, policy subsidizes borrowing.
- Set $\beta = 1$. $y = Ah_{-1}^{0.5}$, $A = 1$ in t_1 and t_3. In t_2, $A = 1$ (no-uncertainty), or $(1.25, 0.75)$ wp $1/2$ (uncertainty). $h = 1$. $m = 1/2$.
  Assume borrowers have initial endowment of h_0 and ω_1 (savers have $1 - h_0$ and $1 - \omega_1$). See how equilibrium changes as h_0 and ω_1 vary (simultaneously)

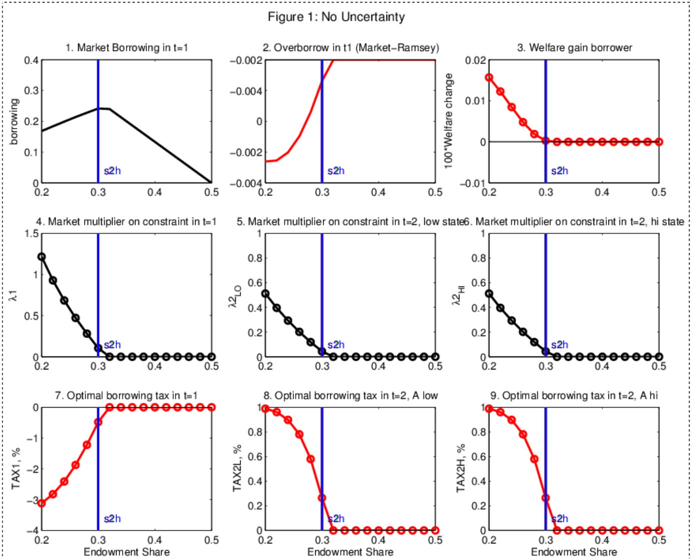
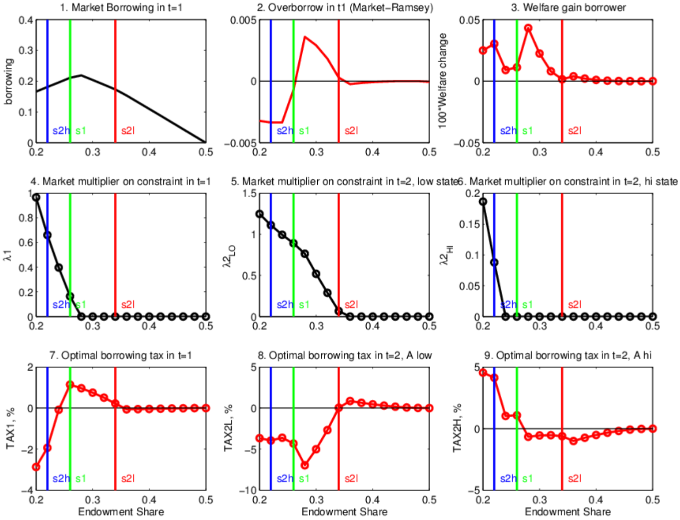


Figure 1: No Uncertainty
1. Market Borrowing in t=1
borrowing
2. Overborrow in t1 (Market−Ramsey)
3. Welfare gain borrower
100*Welfare change
4. Market multiplier on constraint in t=1
λ1
5. Market multiplier on constraint in t=2, low state
λ2LO
6. Market multiplier on constraint in t=2, hi state
λ2HI
7. Optimal borrowing tax in t=1
TAX1, %
8. Optimal borrowing tax in t=2, A low
TAX2L, %
9. Optimal borrowing tax in t=2, A hi
TAX2H, %
s2h
Endowment Share

Figure 2: Uncertainty



A DSGE Model with Multiple Financial Frictions

- Can financial frictions can explain of the quantitative effects of the financial crisis?
- Elements of the Financial crisis
  1. financial institutions suffer losses which impair their ability to extend credit to the real sector, causing a recession.
  2. borrowers balance sheets are impaired, causing a drop in spending
  3. credit supply is tight
- Take to the data a model which embeds these elements
- Model elements: banks and heterogeneous agents
  Event triggering cycles: (1) financial shocks; (2) changes in asset values; (3) changes in credit supply.

Setup

1. **Households.**
   Some households are **Savers**: buy homes, supply deposits D to banking sector and do not face credit constraints.
   Some households are **Borrowers**: borrow L_S against their homes.
2. **Banks** collects deposits from savers and give loans for household borrowers and entrepreneurs.
3. **Entrepreneurs** borrow from bank, transform L into K.
4. Competitive firm rents K and N to produce final good Y.
   HH savers and HH borrowers controlled by wage share in production.
   Size of Entrepreneurs controlled by capital share in production. (some K owned by hh-savers directly, so as to nest RBC as a special case)
5. Shocks: Borrowers subject to **repayment shocks.** Changes in **asset values** and **loan-to-values** affect ability to borrow. The usual suspects (TFP and preference)

Household Savers

Choose consumption, deposits and hours worked

$$\max \sum_{t=0}^{\infty} \beta_H^t \left(A_{p,t} \log C_{H,t} + j A_{j,t} A_{p,t} \log H_{H,t} + \tau \log \left(1 - N_{H,t}\right)\right)$$

s.t.

$$\begin{aligned} & C_{H,t} + \frac{K_{H,t}}{A_{K,t}} + D_t + q_t \Delta H_{H,t} \\ = \quad & \left(R_{M,t} + \frac{1 - \delta_{KH,t}}{A_{K,t}} \right) K_{H,t-1} + R_{H,t-1} D_{t-1} + W_{H,t} N_{H,t}. \end{aligned}$$

Household Borrowers

Low discount factor, creates simple motive for borrowing $\beta_S < \beta_B$ (s stands for subprime)

$$\max \sum_{t=0}^{\infty} \beta_S^t \left(A_{p,t} \log C_{S,t} + j A_{j,t} A_{p,t} \log H_{S,t} + \tau \log \left(1 - N_{S,t}\right)\right)$$

s.t.

$$C_{S,t} + q_t \Delta H_{S,t} + R_{S,t-1} L_{S,t-1} - \varepsilon_t = L_{S,t} + W_{S,t} N_{S,t}$$

$$L_{S,t} \leq E_t \left(\frac{1}{R_{S,t}} m_{S,t} q_{t+1} H_{S,t} \right)$$

If β_S is low enough, constraint on borrowing will hold in a neighborhood of the steady state.
ε_t is the repayment shock

Entrepreneurs

Borrow L_E, .hire N, combine them with K_E, K_H, H_E to produce Y.

$$\max E_0 \sum_{t=0}^{\infty} \beta_E^t \log C_{E,t}$$

s.t.:

$$C_{E,t}+K_{E,t}/A_{K,t}+q_t \Delta H_{E,t}+R_{E,t}L_{E,t-1}+R_{M,t}z_{KH,t}K_{H,t-1}+ac$$
$$= Y_t - WN + \left(1-\delta_{KE,t}\right) K_{E,t-1}/A_{K,t} + L_{E,t} + \varepsilon_{E,t}$$

and

$$L_{E,t} \leq A_{ME,t} \left(m_H \frac{q_{t+1}}{R_{E,t+1}} H_{E,t} + m_K K_{E,t} - m_N \left(W_{H,t} N_{H,t} + W_{S,t} N_{S,t} \right) \right)$$

Borrowing constraint binds if, given R_E, β_E is sufficiently low.
The production function is

$$Y_t = A_{Z,t} \left(z_{KH,t} K_{H,t-1} \right)^{\alpha(1-\mu)} \left(z_{KE,t} K_{E,t-1} \right)^{\alpha\mu} H_{E,t-1}^{\nu} N_{H,t}^{(1-\alpha-\nu)(1-\sigma)} N_{S,t}^{(1-\alpha-\nu)\sigma}$$

Bankers

1. Bankers transform savings into loans. To do so, they are required to hold some equity (bank capital) in their business
2. Bankers are shortsighted: blinded by greed/impatience, they try and borrow as much as they can from household to increase the size of their balance sheet.

Bankers

The banker's problem

$$\max E_0 \sum_{t=0}^{\infty} \beta_B^t \log C_{B,t}$$

where $\beta_B < \beta_H$, subject to:

$$C_{B,t} + R_{H,t-1} D_{t-1} + L_{E,t} + L_{S,t} = R_{E,t} L_{E,t-1} + R_{S,t-1} L_{S,t-1} + D_t - \varepsilon_t$$

ε_t : repayment shock
and additional constraint:

$$D_t \leq \gamma \left(L_{E,t} + L_{S,t} - \varepsilon_t \right) \leftarrow \text{capital adequacy constraint (CAC)}$$

CAC forces banker to hold equity if $\gamma < 1$.

Bank's optimality conditions for deposits and loans:

$$\begin{aligned} 1-\lambda_{B,t} &= E_t\left(m_{B,t}R_{H,t+1}\right) \\ 1-\gamma\lambda_{B,t} &= E_t\left(m_{B,t}R_{E,t+1}\right) \end{aligned}$$

- Expression for spread:

$$E_tR_{E,t+1}-E_tR_{H,t+1}=\frac{\lambda_{B,t}}{m_{B,t}}\left(1-\gamma_E\right).$$

  λ_B : multiplier of bank's capital constraint
  m_B : banker's stochastic discount factor

- Spread is larger when banker's constraint gets tigher (λ_B rises)
- When constraint gets tighter, bank requires larger compensation on loans to be indifferent b/w making loans and issuing deposits. Loans are more illiquid than deposits: when constraint is binding, a reduction in deposits of 1\$ requires cutting back on loans by $\frac{1}{\gamma_E}$\$.
- Rise in spread depresses activity when bank net worth is low.

Remarks

Given the production function

$$Y_t = A_{Z,t} \left(z_{KH,t} K_{H,t-1}\right)^{\alpha(1-\mu)} \left(z_{KE,t} K_{E,t-1}\right)^{\alpha\mu} H_{E,t-1}^{\nu} N_{H,t}^{(1-\alpha-\nu)(1-\sigma)} N_{S,t}^{(1-\alpha-\nu)\sigma}$$

Models becomes an RBC model when

$$\begin{aligned} \mu, \nu &\rightarrow 0: \text{ all capital held by Household Savers} \\ \sigma &\rightarrow 0: \text{ wage share of Household Borrowers is zero} \end{aligned}$$

Alternatively, the model becomes a model without banks (or with frictionless ones) if Household Savers lend directly to Household Borrowers and Entrepreneurs.

Steady State

$$R_H = \frac{1}{\beta_H} \leftarrow \text{return on HH savings}$$

$$\lambda_B = 1 - \beta_B R_H = 1 - \frac{\beta_B}{\beta_H} > 0 \text{ banker is constrained}$$

$$R_E - R_H = (1-\gamma)\left(\frac{1}{\beta_B} - \frac{1}{\beta_H}\right) > 0 \leftarrow \text{spread}$$

Hence $R_E > R_H$ (positive banking spreads):

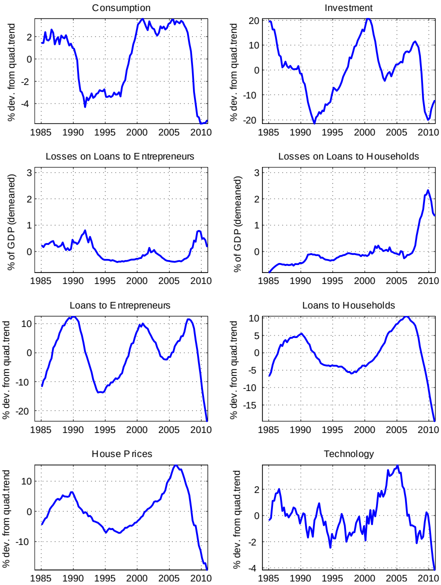
1. Return on bank loans must compensate banker for higher impatience
2. ... must be higher than cost of deposits to make up for higher "liquidity" of loans relative to deposits
   The larger γ, the more loans become substitutes with deposits in the capital adequacy constraint, the lower the extra return on loans required for the bank to be indifferent between borrowing and lending.

Calibration

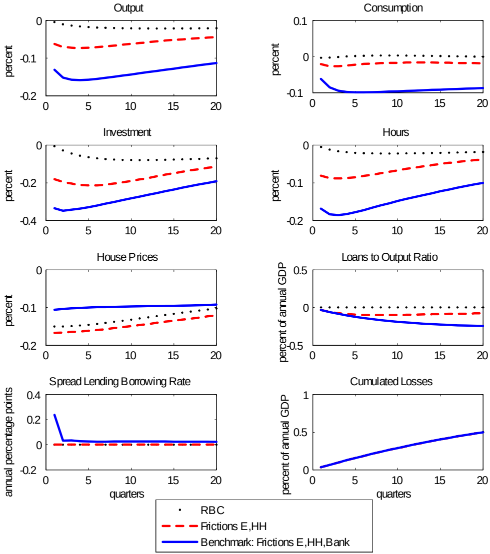
1. Real return on saving 3% per year
2. Capital output ratio about 2, Housing output ratio 1.6
3. m_E, m_S, m_H: 90%, m_K: 50% –> Total nonfinancial sector debt: 105% of GDP
4. For added quantitative realism:
   – inertia in the borrowing and capital adequacy constraint
   – quadratic deposit, loan and capital adjustment costs
   – habits in consumption for all agents
   – variable utilization rate

Bayesian Estimation

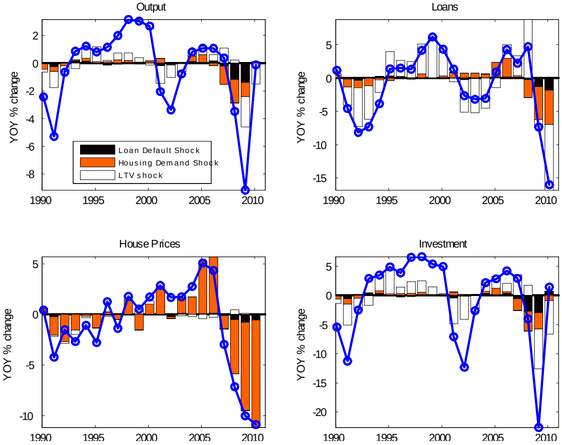
1. Estimated: μ (capital share of constrained entrepreneurs), ν (share of real estate for entrepreneurs), ϕ (adjustment costs), AR(1) process for loan losses ε_t, parameters describing inertia in borrowing and capital adequacy constraints, curvature utilization function, σ, (wage share of constrained HH), η (habit).
2. Use data on
   - Consumption
   - Investment
   - Losses on Loans to Firms
   - Losses on Loans to Households
   - Loans to Households
   - Loans to Firms
   - Housing prices
   - TFP (utilization–adjusted measures from Fernald)
3. 8 shocks (housing demand, repayment shocks for HH and E, LTV shocks for HH and E, preference, investment and TFP shock)
   (3 of the shocks are observed)


Consumption
Investment
Losses on Loans to Entrepreneurs
Losses on Loans to Households
Loans to Entrepreneurs
Loans to Households
House Prices
Technology
% dev. from quad.trend
% of GDP (demeaned)

Financial Shock (blue: baseline model, red: model without banks)



Historical Decomposition



The Timing of the Shocks

1. First stage of financial crisis 2007-2008: Housing Demand Shock drives drop in output
2. Second Stage 2008-2009: Redistribution Shock
3. Third Stage 2009-2010: LTV Shock
   Estimation tells a story in search of a unifying model (and perhaps one single shock): the decline in housing prices causes defaults which in turn cause tighter credit standard.

The Gertler and Karadi's Model

Households

• Within each household, 1 - f "workers" and f "bankers".

• Workers supply labor and return their wages to the household.

• Each banker manages a financial intermediary and also transfers earnings back to
household.

• Perfect consumption insurance within the family.

- Workers become bankers wp $(1-\theta)f$, bankers become workers wp $(1-\theta)$ and transfer wealth to the hh

- HH deposit B_t to the bank/govt debt

$$C_t = W_t L_t + \Pi_t + T_t + R_{t-1} B_{t-1} - B_t$$

Bank Balance Sheet in Gertler and Karadi

$$
\begin{aligned}
Q_t S_t &= N_t + B_t \\
N_t &= R_{kt} Q_t S_{t-1} - R_t B_{t-1} \\
&= \left(R_{kt} - R_t\right) Q_t S_{t-1} + R_t N_{t-1}
\end{aligned}
$$

Bank maximizes PDV of N_t, call it V_t.

- S_t is the quantity of financial claims on non-financial firms that the intermediary holds and Q_t the relative price.
- For the intermediary to operate

$$E_t \beta^i \Lambda_{t,t+1} \underbrace{(R_{kt+i} - R_{ti})}_{\text{excess return}} \geq 0$$

- In turn, if excess return is positive, intermediary would want to expand without bound

Agency Problem

- After the banker/intermediary borrows funds at the end of period t, it may divert the fraction λ of total assets back to its family.
- If the intermediary does not honor its debt, depositers can liquidate the intermediate and obtain the fraction $1 - \lambda$ of initial assets
- Bank subject to incentive constraint

$$V_t > \lambda Q_t S_t$$

- It can be shown that the incentive constraint simplifies to

$$Q_t S_t = \phi N_t$$

  QS : total assets intermediated
  N : total bank capital

- and ϕ_t is the ratio of privately intermediated asset to net worth, bank leverage ratio

Firms

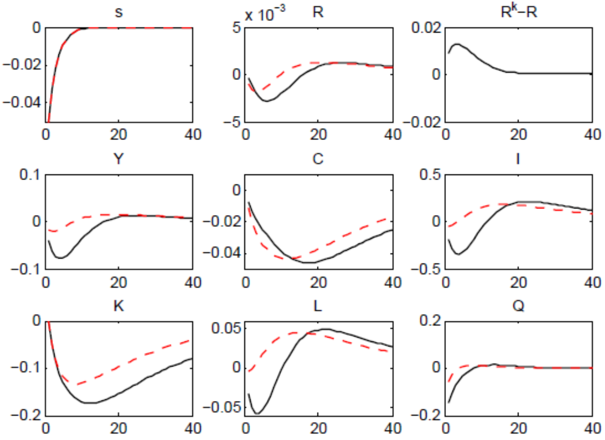
- The firm finances K_t by obtaining funds from intermediaries.
- It issues claims equal to the number of units of capital acquired K_t and prices each claim at the price of a unit of capital Q_t :

$$Q_t K_t = Q_t S_t$$

- Firm combine K with L to produce goods

Comparison for Capital Quality Shock with and without Credit Policy
Credit policy is $Q_t S_t = Q_t \left(S_t^G + S_t^P \right)$ replaces private asset with government assets, thus alleviating constraint

Figure 2: Responses to a Capital Quality Shock



Collateral Constraints and Asymmetries

- How much do Housing Boom and Bust contribute to movements in consumption?
  We address this question with a general equilibrium model estimated with Bayesian methods.
  In the model, housing collateral constraints may bind or not, depending on housing wealth, leverage, and the state of the economy.
- We find that:
  Housing boom 2001-2006: Collateral constraints became slack; the boom contributed little to consumption.
  Housing collapse 2006-2010: Tighter collateral constraints explain three quarters of the fall in consumption.
- Asymmetry is supported by regressions on state- and MSA-level data

The Basic Idea

- Household maximizes $U = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t + j \log h_t\right)$ subject to

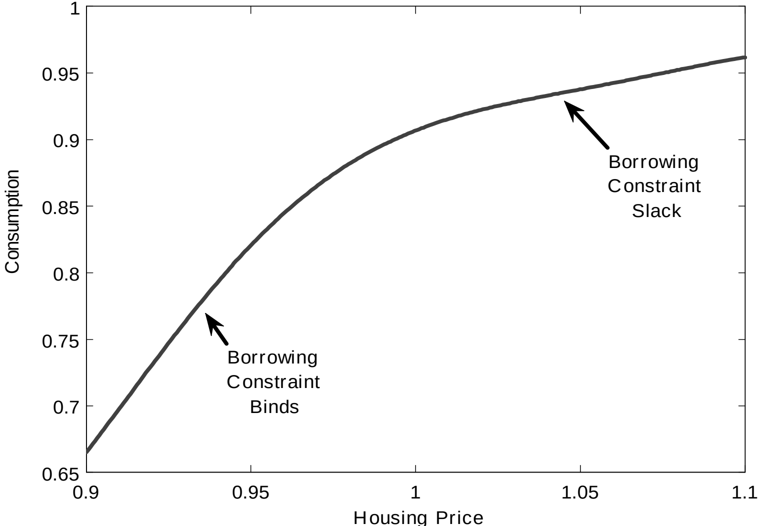
$$\begin{aligned} c_t + q_t h_t &= y + b_t - R b_{t-1} + q_t h_{t-1} (1-\delta) \\ b_t &\leq m q_t h_t \\ \log q_t &= \rho \log q_{t-1} + v_t,\ v_t \sim N\left(0, \sigma^2\right) \end{aligned}$$

- Assume impatience ($\beta R < 1$), fix $y = 1$. The solution of this problem is a consumption function of the form

$$c_t = C\left(q_t, b_{t-1}, h_{t-1}\right)$$

- Consumption function will have the property that consumption increases with house prices, but at a decreasing rate.
  q low –> constraint binds –> consumption moves in lockstep with q
  q high –> constraint is slack –> consumption is less sensitive to q

Model's solution. Consumption function, $c_t = C(q_t, b_{t-1}, h_{t-1})$



The Full Model: Overview

- Standard monetary DSGE model augmented to include housing collateral constraint along the lines of Kiyotaki and Moore (1997), Iacoviello (2005), and Liu, Wang, and Zha (2013).
  Allow for the dual role of housing as a durable good and as collateral for "impatient" households.
- To this framework, add two elements that generate important nonlinearities.
  1. Housing collateral constraint binds only occasionally.
  2. Monetary policy is constrained by ZLB.
- (Monetary DSGE model: RBC core with price and wage rigidities, habits in consumption, and investment adjustment costs)

Full Model: Households and Preferences

Two household types: patient saver and impatient borrower.
Households maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \mathrm{z}_t \left(\Gamma \log \left(c_t - \varepsilon c_{t-1} \right) + \mathrm{j}_t \log h_t - \frac{1}{1+\eta} n_t^{1+\eta} \right),$$

$$E_0 \sum_{t=0}^{\infty} \left(\beta' \right)^t \mathrm{z}_t \left(\Gamma' \log \left(c_t' - \varepsilon c_{t-1}' \right) + \mathrm{j}_t \log h_t' - \frac{1}{1+\eta} n_t'^{1+\eta} \right).$$

z_t : intertemporal preference shock
j_t : housing preference/demand shock

Full Model: Households Constraints

- Patient households maximize their utility subject to:

$$c_t + q_t \Delta h_t + i_t - \frac{R_{t-1} b_{t-1}}{\pi_t} = \frac{w_t n_t}{x_{w,t}} + r_{k,t} k_{t-1} - b_t + div_t,$$

resources are given by wage, capital income, housing wealth, dividends, loan proceeds.

- Impatient households maximize subject to:

$$c'_t + q_t \Delta h'_t + \frac{R_{t-1}}{\pi_t} b_{t-1} = \frac{w'_t}{x'_{w,t}} n'_t + b_t + div'_t,$$

$$b_t \leq \gamma \frac{b_{t-1}}{\pi_t} + (1-\gamma) \, \mathrm{m} q_t h'_t$$

Maximum borrowing b_t = value of house times LTV ratio m
Resources given by wage and housing wealth less loan repayment.
Borrowing constraint allows for inertia, measured by γ

Full Model: Monetary Policy and Supply Side

- Monetary policy follows Taylor rule that responds to annual inflation and GDP in deviation from trend, subject to the zero lower bound (ZLB):

$$R_t = \max\left[1, R_{t-1}^{r_R}\widetilde{\pi}_{a,t}^{(1-r_R)r_\pi}\widetilde{Y}_{t-1}^{(1-r_R)r_Y}\overline{R}^{1-r_R}\mathrm{u}_{r,t}\right].$$

  where $\mathrm{u}_{r,t}$ is an iid monetary policy shock.
- The supply side of the model is completed by production function...

$$Y_t = n_t^{(1-\sigma)(1-\alpha)} n_t'^{\sigma(1-\alpha)} k_{t-1}^{\alpha}$$

- ... and price and wage Phillips curves (derived under Calvo wage and price stickiness).

Two Important Parameters

Parameter	Restriction	Measures
σ	b/w 0 and 1	Collateral Constraints
β'	less than β	Asymmetries

- σ measures wage share of impatient households and importance of collateral constraints.
  When $\sigma = 0$, financial frictions disappear, and the model is a standard monetary DSGE model.
- β' measures the importance of asymmetries.
  With β' very low (relative to β), the asymmetries due to financial frictions become small.

Data and Shocks

- The estimation is based on observations from 1985Q1 to 2011Q4:
  1. Real Total Household Consumption,
  2. Price (GDP deflator) Inflation,
  3. Wage Inflation (compensation per hour, nonfarm),
  4. Real Business Fixed Investment,
  5. Real Housing Prices (Corelogic),
  6. Federal Funds Rate.
- Six shocks – investment-specific shocks, wage markup, price markup, monetary policy, intertemporal preferences, and preferences for housing.

Solution Method

- Solve the model using Occbin
- Depending on whether the zero lower bound binds or not, and depending on whether the collateral constraint binds or not, identify four regimes.
- The solution method links the first-order approximation of the equilibrium conditions describing each regime.
- The dynamics in each regime depend on the expected duration of the regime. In turn, the expected duration depends on the state vector.
- The advantage of the method is its accuracy and speed. Speed is what allows us to compute the model's likelihood in seconds.

Computing the Likelihood

- Model solution can be written as:

$$X_t = \mathbf{P}(X_{t-1}, \epsilon_t) X_{t-1} + \mathbf{D}(X_{t-1}, \epsilon_t) + \mathbf{Q}(X_{t-1}, \epsilon_t)\epsilon_t$$

- In terms of observables, through observation equation $Y_t = \mathbf{H} X_t$,:

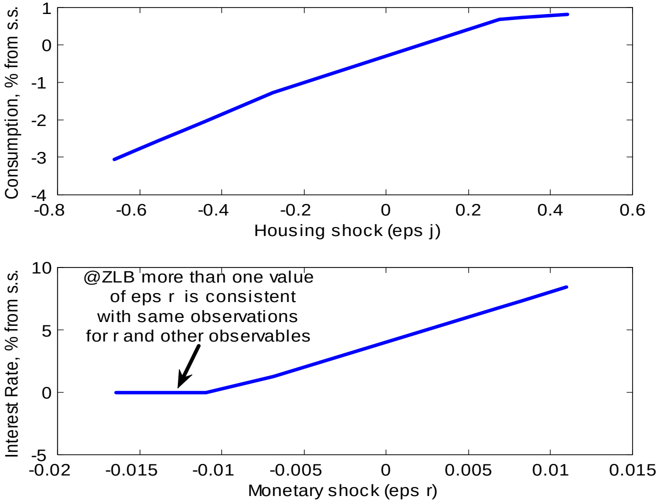
$$Y_t = \mathbf{HP}(X_{t-1}, \epsilon_t) X_{t-1} + \mathbf{HD}(X_{t-1}, \epsilon_t) + \mathbf{HQ}(X_{t-1}, \epsilon_t)\epsilon_t$$

- We initialize X_0 and recursively solve for ϵ_t, given X_{t-1} and current Y_t.
- Given that ϵ_t is $NID(0, ^\circ)$, a change in variables argument implies that the log likelihood for $Y^T \equiv \{Y_t\}_{t=1}^T$ given parameters can be derived analytically as:

$$\log f(Y^T) = -\frac{T}{2}\log(\det{}^\circ) - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t'{}^{\circ -1}\epsilon_t + \sum_{t=1}^{T}\log(|\det\frac{\partial \epsilon_t}{\partial Y_t}|)$$

- where $\frac{\partial \epsilon_t}{\partial Y_t} = inv\left(\mathbf{HQ}_t\right)$ is the Jacobian matrix of the transformation from the shocks to the observations

Does our approach uniquely identify shocks? In practice, yes. Note: exclude interest rate and monetary shocks from observables/shocks when at the ZLB.



Calibration

m	Maximum LTV	0.9
η	labor disutility	1
β	discount factor, patient agents	0.995
$\overline{\pi}$	steady-state gross inflation rate	1.005
α	capital share in production	0.3
δ	capital depreciation rate	0.025
$\bar{j}$	housing weight in utility	0.04
X_p, X_w	average price and wage markup	1.2

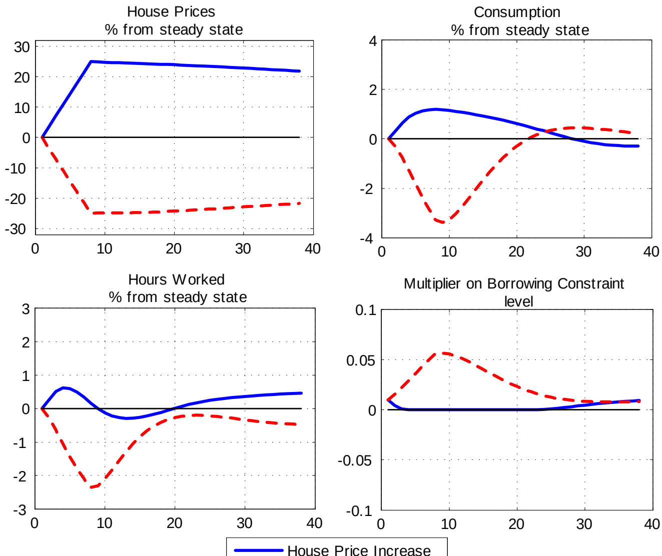
We estimate:

- the parameters governing the shocks processes;
- the parameters governing the nominal and real rigidities;
- the parameters governing the monetary policy rule;
- the wage share of impatient households, and their discount rate.

Model Results: Selected Estimated Parameters

		Prior type [mean, std]	Posterior Mode
β'	discount factor, impatients	normal [0.99, .0015]	**0.9895**
σ	wage share, impatients	beta [0.5, 0.20]	**0.4151**
ε	habit in consumption	beta [0.5, 0.1]	**0.6399**
ϕ	investment adjustment cost	gamma [5, 2]	**5.0307**
r_π	inflation resp. Taylor rule	normal, 1.5, 0.25]	**1.7385**
r_R	inertia Taylor rule	beta [0.75, 0.1]	**0.5200**
r_Y	output response Taylor rule	beta [0.125, 0.025]	**0.0796**
θ_π	Calvo parameter, prices	beta [0.5, 0.075]	**0.9190**
θ_w	Calvo parameter, wages	beta [0.5, 0.075]	**0.9170**
γ	inertia borrowing constraint	beta [0.5, 0.1]	**0.4547**

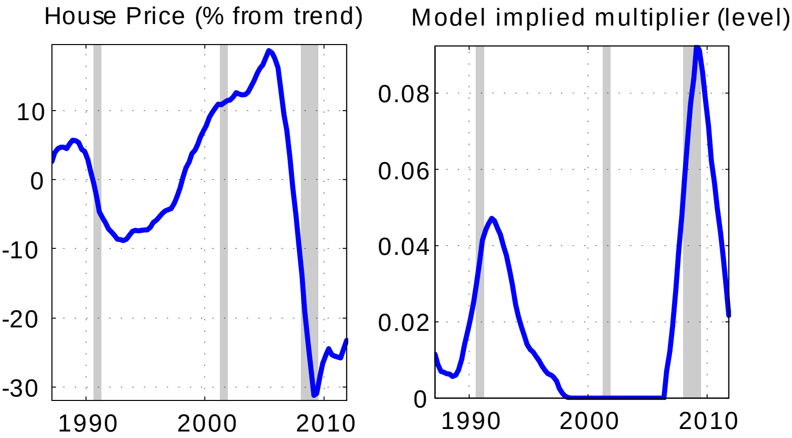
Housing demand shocks around steady state



Consumption and the Housing Boom and Bust

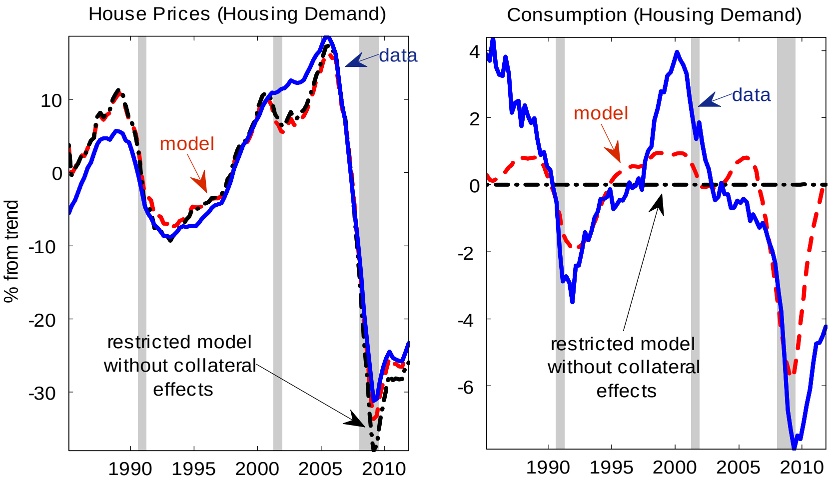
- By construction, estimated model explains everything in sample. However, it is important to study which shocks and frictions are important in driving the model's dynamics.
- To understand the importance of collateral constraints, estimate the restricted model with $\sigma = 0$, and run a horse race between baseline model and model with $\sigma = 0$.

The estimated simulated multiplier



Consumption and House Prices: Data and Model

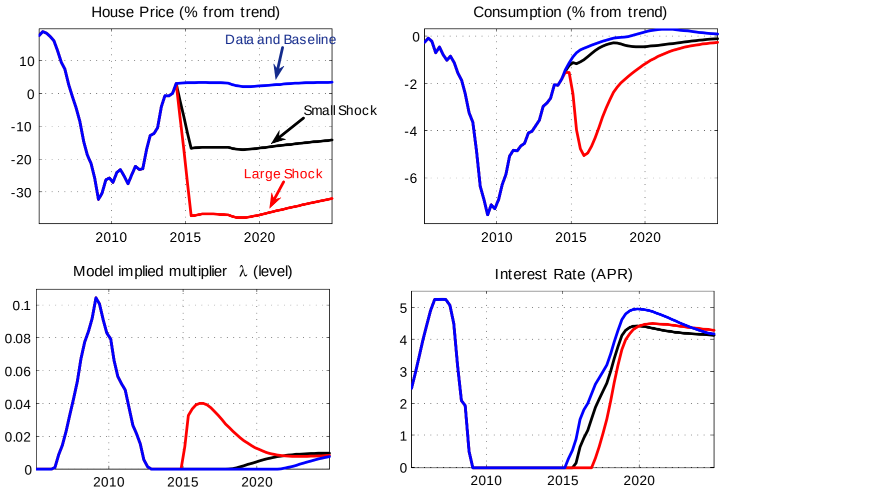
(Housing Demand Shocks Only - model w/o frictions)



Summary of Comparison

- The baseline model comes close to matching the evolution of both housing and consumption with just the housing shocks,
- By contrast, housing shocks have no bearing on consumption for the model without the collateral constraints.
- Restricted model is completely dependent on a sequence of large consumption shocks to match the consumption data.
- A posterior odds ratio of 90 to 1 favors the baseline model that does not call for the additional sequence of consumption shocks.

Asymmetry in Action



Intro: A Baby Example

We have $n = 5$ independent observations from the normal distribution for the variable z_t, mean zero (known) and unit standard deviation (known).

$$z = [-0.5925, 0.3298, -0.9984, 1.8028, -0.5416]$$

What is the likelihood of observing this sample? From the formula for the density of an $n-$variate normal distribution with mean 0 and variance covariance matrix Σ, we have

$$L = (2\pi)^{-T/2} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}z'\Sigma^{-1}z\right]$$

$$L = (2\pi)^{-5/2} \exp\left(-\frac{0.5925^2 + 0.3298^2 + 0.9984^2 + 1.8028^2 + 0.5416^2}{2}\right) = .00082948$$

$$\ln L = \ln 8.2948 \times 10^{-4} = -7.0947$$

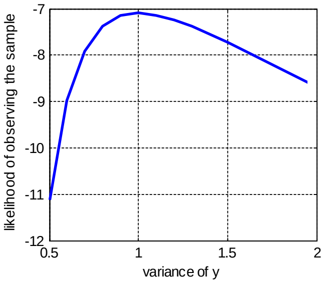
Estimation of DSGE models requires knowing (or remembering) all of the above, plus a lot of practice, and a few other things.

Why is it called Estimation? We did not Estimate Anything...

Treat the likelihood as a function of Σ. Then can write it as

$$L = L\left(z_T, \Sigma\right)$$

L : likelihood of observing particular sequence z_T as function of the parameter Σ.
Estimate of Σ is the value that maximizes the likelihood function above.
In this case, the likelihood function is the plot to left. Most often, with DSGE models, you end up with a likelihood that looks like the one on the right.



General Problem

1. Consider solution of a DSGE model

$$\begin{aligned} x_t &= F(\mu)\, x_{t-1} + G(\mu)\, v_t \\ z_t &= H' x_t \end{aligned}$$

   $E(Gvv'G') = Q$. F and G are functions of vector of model parameters μ. $\dim(x) = m \times 1$, $\dim(z) = n \times 1$, $\dim(H) = m \times n$. Let $P_{t|t-1} \equiv E\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)'$
2. We are interested in estimating unknown parameters in vector μ based on a sample observations about $z^T \equiv \{z_t\}_{t=1}^T$
3. Define as ML estimates of the model the values of μ that maximize the likelihood associated with a particular sample of realizations of z over time.
4. Likelihood associated with particular realization of z at time t as $L\left(z_t|z^{t-1}\right)$. Sequence of conditional likelihoods $\left\{L\left(z_t|z^{t-1}\right)\right\}_{t=1}^T$ is independent over time, thus

$$L\left(z^T\right) = \Pi_{t=1}^T L\left(z_t|z^{t-1}\right)$$

5. $L\left(z^T\right)$ is the likelihood of our model. But, how do we compute $L\left(z_t|z^{t-1}\right)$?

Writing Down the Likelihood: All Variables are Observed (1)

- All the variables in x are "observed" ($z = x$). In that case (provided #shocks=#observables), calculation of the likelihood proceeds as in standard econometric textbooks.
- Conditional on $\left\{x_j\right\}_{j=1}^{t-1}$, note that the optimal forecast of x_t is given by

$$\widehat{x}_t = F x_{t-1}$$

so that the error in predicting x_t is

$$\widehat{e}_t = x_t - F x_{t-1}$$

conditional likelihood associated with a realization of x_t can be assessed as the likelihood assigned to $\widehat{e}_t$ by its pdf

$$L\left(x_t | x^{t-1}\right) = p_e\left(\widehat{e}_t\right)$$

Writing Down the Likelihood: All Variables are Observed (2)

- The likelihood evaluation begins by inserting x_1 into its unconditional distribution, which is $N(0, Q)$, hence

$$L(x_1|\mu) = (2\pi)^{-m/2} \left|Q^{-1}\right|^{1/2} \exp\left[-\frac{1}{2}\left(x_1' Q^{-1} x_1\right)\right]$$

then, for $t = 2, ..., T$, we have

$$L(x_t|\mu) = (2\pi)^{-m/2} \left|Q^{-1}\right|^{1/2} \exp\left[-\frac{1}{2}(x_t - Fx_{t-1})' Q^{-1} (x_t - Fx_{t-1})\right]$$

- Finally the sample likelihood is the product of the individual likelihoods.

$$L(x|\mu) = \Pi_{t=1}^{T} L(x_t|\mu)$$

Writing Down the Likelihood: Some Variables are not Observed(1)

- The more complicated case is when $n < m$. In this case, one need to find a way to infer the value of x from observations on z. Inferring the values of x is essential to calculate the likelihood of z.
- To do so, we use in practice a particular algorithm (the Kalman filter) which is used to produce assessment of the conditional probability $L\left(z_t|z^{t-1}\right)$ associated with the time-t observation z_t, given the history of past realizations $z^{t-1} \equiv \left\{z_j\right\}_{j=1}^{t-1}$.
- Hidden states and observables are described by a state space system that is perturbed at each point by Gaussian shocks with zero mean and known covariances.
- The next slides sketch a recursive version of the Kalman filtering problem that allows computing recursively the likelihood of a model.

Writing Down the Likelihood: Some Variables are not Observed (2)

Computation is recursive: start at time 0, where we can calculate the following:

- $z_{1|0}$: Conditional expectation of z_1 (the vector of unobservables that enter the computation of the likelihood) given observations on z_0
- z_1: Actual observations on z_1
- $x_{1|0}$: Conditional expectation of x_1 given observations on z_0
  At time zero, we want to derive the best estimate of x_1.
- The key question is how to estimate x_1 given $x_{1|0}$ and z_1.

  Using $x_t = Fx_{t-1} + Gv_t$ and $P_{t|t-1} \equiv E\left(\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)'\right)$

$$
\begin{aligned}
x_{1|0} &= 0 \\
P_{1|0} &= FP_{1|0}F' + Q
\end{aligned}
$$

- This way, one can construct associated values for observables z, given by:

$$
\begin{aligned}
z_{1|0} &= H'x_{1|0} = 0 \\
\Omega_{1|0} &= E\left[\left(z_1 - z_{1|0}\right)\left(z_1 - z_{1|0}\right)'\right] = H'P_{1|0}H \rightarrow \text{innovation covariance}
\end{aligned}
$$

Writing Down the Likelihood: Some Variables are not Observed (3)

- The two objects above are used to compute the likelihood function of z_1, which is a normal variable with mean $z_{1|0}$ and variance $\Omega_{1|0}$, that is $z_1 \tilde{} N\left(z_{1|0}, \Omega_{1|0}\right)$

$$L\left(z_1|\mu\right) = \left(2\pi\right)^{-m/2} \left|\Omega_{1|0}^{-1}\right|^{1/2} \exp\left[-\frac{1}{2}\left(z_1' \Omega_{1|0}^{-1} z_1\right)\right].$$

- Next, the values of $x_{1|0}$ and $P_{1|0}$ are updated to construct new updates of $x_{1|1} \equiv x_1$ and $P_{1|1} \equiv P_1$.

$$\begin{aligned} x_{1|1} &= \underbrace{x_{1|0}}_{\text{old value}} + \underbrace{P_{1|0} H \Omega_{1|0}^{-1}}_{\text{Kalman gain}} \underbrace{\left(z_1 - z_{1|0}\right)}_{\text{prediction error}} \rightarrow \text{updated state estimate} \\ P_{1|1} &= P_{1|0} - P_{1|0} H \Omega_{1|0}^{-1} H' P_{1|0} \rightarrow \text{updated covariance estimate} \end{aligned}$$

Writing Down the Likelihood: Some Variables are not Observed (4)

The term $K_1 \equiv P_{1|0} H \Omega_{1|0}^{-1}$ denotes the Kalman gain matrix. It is a minimum-mean square estimator that yields the best prediction of x_1 given estimates of $x_{1|0}$, z_1 and $z_{1|0}$. It can be derived as follows; consider the covariance matrix of x_1, $P_{1|1}$

$$P_{1|1} = E\left[\left(\underset{\text{actual}}{x_1} - \underset{\text{mean}}{x_{1|1}} \right) \left(x_1 - x_{1|1} \right)' \right]$$

$$\text{use } x_{1|1} = x_{1|0} + K_1 \left(z_1 - z_{1|0} \right) \text{ for some } K_1 \text{ to be determined}$$

$$P_{1|1} = cov\left(x_1 - x_{1|0} - K_1 \left(z_1 - z_{1|0} \right) \right) = P_{1|1} = cov\left(x_1 - x_{1|0} - K_1 H' \left(x_1 - x_{1|0} \right) \right)$$

$$P_{1|1} = E\left[\left(I - K_1 H' \right) P_{1|0} \left(I - K_1 H' \right)' \right]$$

Minimize the expected value of the square of the magnitude of this vector.

$$\min_{K_1} trace\left(P_{1|1} \right) \rightarrow K_1 = P_{1|0} H \Omega_{1|0}^{-1}$$

Writing Down the Likelihood: Some Variables are not Observed (4)

Next, for every other period $t = 2, ..., T$, we have:

$$\begin{aligned}
x_{t|t-1} &= Fx_{t-1} \\
P_{t|t-1} &= FP_{t|t-1}F' + Q \\
z_{t|t-1} &= H'x_{t|t-1} \\
\Omega_{t|t-1} &= H'P_{t|t-1}H \\
L\left(z_t|\mu\right) &= (2\pi)^{-m/2}\left|\Omega_{t|t-1}^{-1}\right|^{1/2}\exp\left[-\frac{1}{2}\left(\left(z_t - z_{t|t-1}\right)'\Omega_{t|t-1}^{-1}\left(z_t - z_{t|t-1}\right)\right)\right] \\
x_{t|t} &= x_{t|t-1} + P_{t|t-1}H\Omega_{t|t-1}^{-1}\left(z_t - z_{t|t-1}\right) \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}H\Omega_{t|t-1}^{-1}H'P_{t|t-1} \\
L\left(z|\mu\right) &= \Pi_{t=1}^{T}L\left(z_t|\mu\right)
\end{aligned}$$

Bayesian Estimation

- **Classical estimation**: parameters treated as fixed but unknown, and likelihood function is interpreted as sampling distribution from data. Realizations of z interpreted as one possible realizations from $L(z|\mu)$. Inferences on μ are statements regarding probabilities associated with particular realizations of z given μ.
- **Bayesian estimation**: observations on z treated as given. Make inferences about distribution of μ conditional on z. Probabilistic interpretation of μ allows incorporating judgements on μ through prior distribution $\pi(\mu)$.
- From the definition of joint probability, we have that:

$$p(\mu, z) = L(z|\mu)\pi(\mu)$$

reversing the role of μ and z gives

$$p(z, \mu) = P(\mu|z)\,p(z).$$

Solving for $P(\mu|z)$ gives

$$\underset{\text{posterior}}{P(\mu|z)} = \frac{L(z|\mu)\pi(\mu)}{p(z)} \propto \underset{\text{likelihood}}{L(z|\mu)}\ \underset{\text{prior}}{\pi(\mu)}$$

where $p(z)$ is a constant from the point of view of the distribution for μ

Applications to DSGE Models

We consider three examples of estimation. Dynare files are on the course webpage.

1. The toy model
2. A model with a Phillips curve (identification)
3. A richer dynamic-new-keynesian model

Do not attempt anything more complicated than this until you have fully mastered these examples.

1. Toy Model

- The model is described by

$$y_t = e_t$$

  where e_t is an exogenous iid shock with zero mean and unknown variance σ^2. We want to estimate σ. In principle, we can impose a prior on the distribution of σ and combine it with information from the data on y.

- See `basic_estimation`

2. Phillips curve

- Model is

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + k y_t \\ y_t &= \rho y_{t-1} + e_t, \;\; e_t \sim N\left(0, \sigma^2\right) \end{aligned}$$

- Assume our only obervable is π_t. The solution to our model takes the form

$$\begin{aligned} x_t &= F x_{t-1} + G v_t \\ z_t &= H' x_t \end{aligned}$$

$$\begin{aligned} x_t &= \begin{bmatrix} \pi_t & y_t \end{bmatrix}', z_t = [\pi_t], v_t = [e_t] \\ F &= \begin{bmatrix} 0 & \frac{\kappa\rho}{1-\beta\rho} \\ 0 & \rho \end{bmatrix}, G = \begin{bmatrix} \frac{\kappa}{1-\beta\rho} \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

- Note the rational expectations solution for π_t

$$\pi_t = \rho \pi_{t-1} + \frac{\kappa}{1-\beta\rho} \varepsilon_t$$

- Estimation will recover ρ, and only one parameter among β, σ_ε and κ. ML will fail to recover separately estimates of $\kappa, \beta, \sigma_\varepsilon$. (see `toy_pc`)

3. New Keynesian Model

- Consider the following log-linear model for output y, inflation π and the nominal interest rate R. iid shocks are g and u

$$\begin{aligned} y_t &= E_t y_{t+1} - R_t + E_t \pi_{t+1} + g_t \\ \pi_t &= k y_t + \beta E_t \pi_{t+1} + u_t \\ R_t &= \phi \pi_t \end{aligned}$$

- Solution is (`toy_dnk`)

$$y_t = \frac{1}{1+\phi k} g_t - \frac{\phi}{1+\phi k} u_t, \quad \pi_t = \frac{k}{1+\phi k} g_t + \frac{1}{1+\phi k} u_t$$

- Estimation cannot recover β, and at most three parameters among σ_g, σ_u, ϕ and k. To think about why, the series are iid, and all that enters the likelihood function is their variance and their covariance.

$$var\left(y\right) = \frac{1}{\left(1+\phi k\right)^2} \sigma_g^2 + \frac{\phi^2}{\left(1+\phi k\right)^2} \sigma_u^2; \quad var\left(\pi\right) = \frac{k^2}{\left(1+\phi k\right)^2} \sigma_g^2 + \frac{1}{\left(1+\phi k\right)^2} \sigma_u^2$$

$$cov\left(y, \pi\right) = \frac{k}{\left(1+\phi k\right)^2} \sigma_g^2 - \frac{\phi}{\left(1+\phi k\right)^2} \sigma_u^2$$

MCMC

- Key computational problem: how to compute the distribution of μ, $P(\mu|z)$: standard Monte Carlo integration techniques cannot be used, because one cannot draw random numbers directly from $P(\mu|z)$.
- Typically, we use Markov Chain Monte Carlo (MCMC) techniques, in particular the Metropolis-Hastings algorithm which is a particular version of the MCMC algorithm. The idea of the algorithm is to explore the distribution and to weigh to outcomes appropriately.

See Chapter 9 in Dejong and Dave for more details.

Basic Idea of the Kalman Filter

- Consider the following system

$$\begin{aligned} s_t &= Fs_{t-1} + G\omega_t, \omega_t \sim N(0, Q) && \text{(state\_equation)} \\ y_t &= Hs_t + v_t, v_t \sim N(0, R) && \text{(measurement\_equation)} \end{aligned}$$

  y_t is observed, everything else is not

- Problem: want to compute the following objects

  likelihood function of y^T

  recover from the y^T the sequences of s^T,ω^T,v^T

- Note that the above implies

$$y_t = HFs_{t-1} + HG\omega_t + v_t$$

  in order to compute y_t, we need to know s_{t-1}, ω_t, v_t
  The Kalman filter offers a way to go compute s, ω, v, and therefore to compute the likelihood of y^T.

- Inputs $(y_t, H, F, G) - - >$ Output $(\omega^T, s^T, v^T, Q, R)$

The Workings of the filter

- Suppose we are in t, and know s^{t-1} as well as its covariance matrix Σ_{t-1}. At that point, we are interested in computing an estimate of s_t given surprises in y_t

$$s_t = E_{t-1}s_t + K_t\left(y_t - E_{t-1}y_t\right)$$

- Note that

$$s_t = E_{t-1}s_t + K_t\left(y_t - HE_{t-1}s_t\right)$$

- Think of the best K_t as solving a minimization problem. It tells you how much you want to change your estimate given a measurement. Intuitively, the more y_t changes, the more likely it is that you want to change s_t (K_t large)
- So, how do we derive K_t

Deriving K

- Start with estimates of $E_{t-1}\Sigma_t$, $E_{t-1}s_t$, together with y_t. Then

$$
\begin{aligned}
K_t &= E_{t-1}\Sigma_t H' \left(HE_{t-1}\Sigma_t H' + R\right)^{-1} \\
\Sigma_t &= E_{t-1}\Sigma_t - K_t H E_{t-1}\Sigma_t \\
s_t &= E_{t-1}s_t + K_t \left(y_t - HE_{t-1}s_t\right) \\
E_t\Sigma_{t+1} &= F\Sigma_t F' + GQG' \\
E_t s_{t+1} &= F s_t
\end{aligned}
$$

- At this point, we are in $t+1$ and we can start again.
- Typically $E_{t-1}\Sigma_t = \Sigma^*$, $E_{t-1}s_t = \bar{s}$
- The formula for K_t is the outcome of a minimization problem where we try to minimize the squared forecast error of s_t

A DSGE Model of the Housing Market

- Two questions:

1. What is the nature of the shocks hitting the housing market?
2. How big are spillovers from the housing market to the wider economy?

- To answer them, build and estimate a quantitative model with:
  - nominal rigidities and monetary policy;
  - multi-sector structure with housing;
  - financing frictions on the household side.

Housing and the Macroeconomy

	Household balance sheet, 2008 billion $		FOF entry
A	**Assets**	**67,134**	B100:1
B	Real Estate (Owner-Occupied Homes)	20,398	B100:3
C	Residential Real Estate of Noncorporate Business (Rented Homes)	4,964	B103:4
D	Other Tangible Assets	4,779	B100:2 less B100:3
E	Financial Assets less Residential Real Estate of Noncorp. Business	36,992	B100:8 less B103:4
F	**Liabilities**	**14,216**	B100:31
H	**Household net worth**	**52,917**	A-F
	Housing wealth	25,362	B+C
	Non housing wealth	27,555	D+E-F

Table 1.1: Composition of Household Wealth in the United States

Model

- Production
  - $Y-$sector produces C, IK, intermediate goods (using K and N)
  - $IH-$sector produces new homes (using K, N, land and interm. goods)
  - Different trend progress across sectors (C, IK, IH)
- Households
  - Patients work, consume, buy homes, rent capital and land to firms and lend to impatient households
  - Impatients/Credit Constrained work, consume, buy homes and borrow against value of home
    (We set up preferences in a way that borrowing constraint is binding)
- Sticky prices in the non-housing sector, sticky wages in both sectors
- Real rigidities: habits in C, imperfect labor mobility, K adjustment costs, variabile K utilization

Firms

- Firms produce :

$$Y_t = (A_{ct} N_{ct})^{1-\mu_c} (z_{ct} k_{ct-1})^{\mu_c}$$
$$IH_t = (A_{ht} N_{ht})^{1-\mu_h-\mu_b-\mu_l} (z_{ht} k_{ht-1})^{\mu_h} k_{bt}^{\mu_b} l_{t-1}^{\mu_l}.$$

- Y_t : non-housing, sticky price sector,
  IH_t flex price sector
- Two types of households/workers of measure 1
  α : wage share of unconstrained households (lenders)
  $1-\alpha$: wage share of constrained households (borrowers)

$$N_c = n_c^{\alpha} n_c'^{1-\alpha}, N_h = n_h^{\alpha} n_h'^{1-\alpha}$$

Lenders

$$\max E_0 \sum_{t=0}^{\infty} (\beta G_C)^t z_t \left(\log \widetilde{c}_t + j_t \log h_t - \tau_t g\left(n_{ct}, n_{ht}\right)\right)$$

- subject to budget constraint:

$$c_t + \frac{k_{ct}}{A_{kt}} + k_{ht} + q_t \left(h_t - (1-\delta_h)\, h_{t-1}\right) + b'_t$$

$$= \widetilde{R}_{ct} k_{ct-1} + \widetilde{R}_{ht} k_{ht-1} + R_{lt} l_{t-1} + Div_t + wage'_t + \frac{R_{t-1} b'_{t-1}}{\pi_t}$$

Borrowers

- Discount future more heavily ($\beta' < \beta$)

$$\max E_0 \sum_{t=0}^{\infty} \left(\beta' G_C\right)^t z_t \left(\log \tilde{c}'_t + j_t \log h'_t - \tau_t g\left(n'_{ct}, n'_{ht}\right)\right)$$

- subject to budget constraint

$$c'_t + q_t \left(h'_t - (1-\delta_h) h'_{t-1}\right) = wage'_t + b'_t - \frac{R_{t-1}}{\pi_t} b'_{t-1}$$

- and to borrowing constraint

$$b'_t \leq m E_t \left(q_{t+1} h'_t \pi_{t+1} / R_t\right)$$

m : loan-to-value ratio

Monetary Policy

$$R_t = (R_{t-1})^{r_R} \left(\pi_t^{r_\pi} \left(\frac{GDP_t}{G_C GDP_{t-1}} \right)^{r_Y} \overline{rr} \right)^{1-r_R} \frac{\mathsf{u}_{Rt}}{\mathsf{s}_t}$$

u_{Rt} : iid monetary policy shock

s_t : persistent inflation objective shock

In Guerrieri and Iacoviello (2013), allow for ZLB

Shocks

- Stationary AR(1)
  z_t : preference (discount factor) shock
  j_t : housing demand shock (*or household technology shock*)
  τ_t : labor supply shock
  u_{Rt} : monetary shock (iid)
  s_t : inflation objective shock
  u_{pt} : markup/inflation shock (iid)
- Trend-stationary shocks

$$
\begin{aligned}
\ln A_{ct} &= t\ln(1+\gamma_{AC}) + \ln Z_{ct}, \quad \ln Z_{ct} = \rho_{AC}\ln Z_{ct-1} + u_{Ct} \\
\ln A_{ht} &= t\ln(1+\gamma_{AH}) + \ln Z_{ht}, \quad \ln Z_{ht} = \rho_{AH}\ln Z_{ht-1} + u_{Ht} \\
\ln A_{kt} &= t\ln(1+\gamma_{AK}) + \ln Z_{kt}, \quad \ln Z_{kt} = \rho_{AK}\ln Z_{kt-1} + u_{Kt}
\end{aligned}
$$

Model Workings

1. At a basic level, it works like an RBC model with sticky prices/wages in the $Y-$sector, like an RBC with flex prices/sticky wages in the $IH-$sector (added twist: IH sector produces durables)
2. Sector specific shocks or preference shocks can shift resources from one sector to the other
3. Housing collateral generates wealth effects on consumption

Trends

1. Log preferences and Cobb-Douglas yield balanced growth
2. C and qIH grow at the same rate over time.
3. IK can grow faster than C, thanks to A_K progress
4. IH can grow slower than C, if land is a limiting factor and A_H is slow
5. Long-run growth rates

$$\begin{aligned}
\frac{\Delta C}{C} &= \gamma_{AC} + \frac{\mu_c}{1-\mu_c}\gamma_{AK} \\
\frac{\Delta IK}{IK} &= \gamma_{AC} + \frac{1}{1-\mu_c}\gamma_{AK} \\
\frac{\Delta IH}{IH} &= (\mu_h + \mu_b)\,\gamma_{AC} + \frac{\mu_c\,(\mu_h+\mu_b)}{1-\mu_c}\gamma_{AK} + (1-\mu_h-\mu_l-\mu_b)\,\gamma_{AH} \\
\frac{\Delta q}{q} &= (1-\mu_h-\mu_b)\,\gamma_{AC} + \frac{\mu_c\,(1-\mu_h-\mu_b)}{1-\mu_c}\gamma_{AK} \\
&\quad -(1-\mu_h-\mu_l-\mu_b)\,\gamma_{AH}
\end{aligned}$$

2. Estimation

1. Use 10 time-series (1965Q1-2006Q4) for US
   per capita C, IH, IK, real housing price q
   R, π, sectoral hours N_c and N_h, sectoral wages Δw_c and Δw_h
2. Some parameters calibrated to match steady state ratios
   $\beta = 0.9925$, $\beta' = 0.97$, $m = 0.85$
   $Y = N_c^{0.65} k_c^{0.35}$, $IH = N_h^{0.70} k_h^{0.10} k_b^{0.10} l^{0.10}$
   Targets: $(K + qH)\,/GDP = 3.2$, $(qH)\,/GDP = 1.35$, $(\delta_h qH)\,/GDP = 0.06$
3. Other parameters (including degree of financing frictions) estimated by Bayesian techniques

3. Results

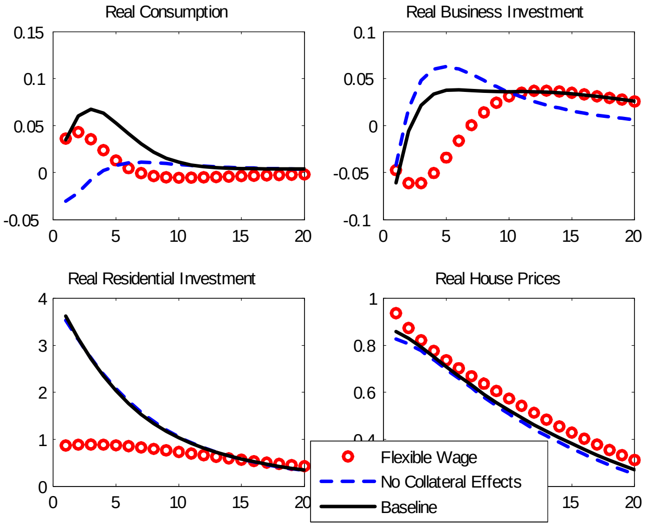
Prior and Posterior Parameters

1. Slow rate of technological progress in housing construction $(\gamma_{AC} = 0.32\%,\ \gamma_{AH} = 0.08\%)$
2. Wage share of credit constrained households $1 - \alpha = 21$ percent
3. High price rigidity $(\theta_\pi = 0.83)$ and indexation $(\iota_\pi = 0.71)$
   High wage rigidity $(\theta_{wc} = 0.81, \theta_{wh} = 0.91)$, low wage indexation $(\iota_{wc} = 0.07, \iota_{wh} = 0.42)$
4. Taylor rule: $R_t = 0.61 R_{t-1} + 0.39\left[1.38\pi_t + 0.51\left(gdp_t - gdp_{t-1}\right)\right]$

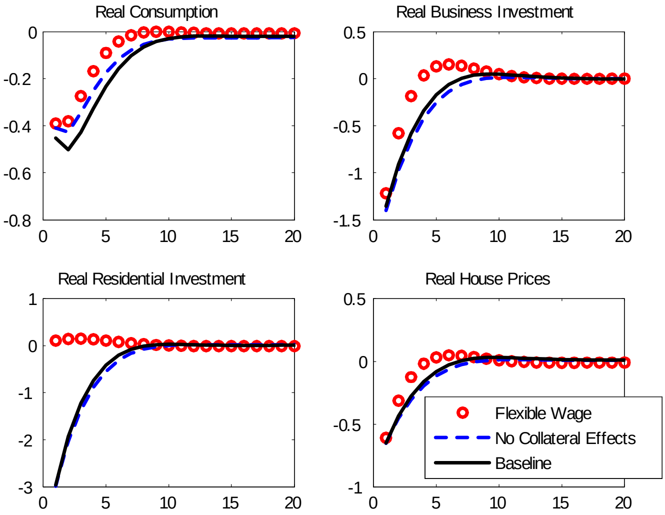
Variance Decomposition

Housing demand shocks and housing technology shocks account for one quarter each of the volatility of residential investment and house prices.
Monetary shocks account for about 20 percent

Impulse Responses, Housing Preference Shocks



Impulse Responses, Monetary Shocks



Role of Monetary Shocks

1. Sensitivity of residential investment to monetary shocks larger than that of business investment, in line with VAR evidence
2. Key reason: wage stickiness
   If IH sector were flex wage, flex price, it would not contract after contractionary policy (BHK 2007)
3. Model elasticity of house prices to a monetary shocks of similar magnitude to what is found in VAR studies

Our two original questions, revisited.

1. What drives the housing market? Focus on recent period.
2. How big are the spillovers? Focus on pre and post 1980's

Drivers of Housing Market

Focus on 2000-2006:

Period		% q	Technology	Monetary Pol.
1998:I	2005:I	14.1	5.9	2.1
2005:II	2006:IV	-0.3	-0.2	-2.7
		% IH		
1998:I	2005:I	22.2	-4.1	9.8
2005:II	2006:IV	-15.5	-4.3	-11.4

Comparison with 1976-1985 period: monetary policy has played a larger role here.

Size of Spillovers

- Most of the spillovers are through the effect on consumption.
  For given LTV m, they are a function of α.
  Regression based on artificial data generated by the model

$$\Delta \log C_t = 0.0041 + 0.123 \Delta \log HW_{t-1} \text{ if } \alpha = 0.79$$
$$\Delta \log C_t = 0.0041 + 0.099 \Delta \log HW_{t-1} \text{ if } \alpha = 1$$

- To better measure spillovers in sample, re-estimate the model across subsamples (1965-1982, 1989-2006).
  First period: fix $m = 0.775$, $1 - \widehat{\alpha} = 0.33$
  Second period: fix $m = 0.925$, $1 - \widehat{\alpha} = 0.21$
- Two implications
  Monetary policy more "powerful" in the second period
  Housing shocks have larger spillover effects on consumption in the second period